Total No. of printed pages = 6

63/2 (SEM-2) MAT 204

2022

MATHEMATICS

(Theory Paper)

Paper Code : MAT 204

(General Topology)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. (a) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Show that $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. 5

   (b) Let $(X, \mathcal{T})$ be a topological space and E be a non-empty subset of X. Show that the characteristic function of E on X is continuous if and only if E is both open and closed subset of X. 5

[Turn over

Or

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 2, & \text{if } x \geq 0 \\ -2, & \text{if } x \leq 0 \end{cases}$$

Check whether

(i) f is U – U is continuous.

(ii) f is T – T is continuous.

Where U and T are respectively the usual and lower linit topology on $\mathbb{R}$. 2½+2½=5

(c) Let X be an infinite set and $f: \mathbb{P}(X) \to \mathbb{P}(X)$ be a function defined by

$$f(A) := \begin{cases} A, & \text{if A is finite.} \\ X, & \text{if A is infinite.} \end{cases}$$

Show that f satisfies the Kuratowski closure axioms. Hence find the topology induced by f. 3+2=5

(d) Let f be a continuous open function from a topological space (X, T) onto another space (Y, T*). Show that T* is the quotient topology on Y relative to f. 5

Or

Let f be a continuous closed function from a topological space (X, T) onto another space (Y, T*). Show that T* is the quotient topology on Y relative to f. 5

2\. (a) Let (X, T) be a second countable space with X is uncountable set. Prove that every uncountable subset of X has a limit point. Does the result is true for separable space ? 4+1=5

(b) Show that a topological space is T_1 if and only if it contains the co-finite topology. 5

Or

Let (X, T) be a T_1-space. Show that the derived set of a finite set in X is a null set. 5

(c) Prove or disprove that in a $T_{3½}$ – space any two distinct points can be separated by continuous functuion. 5

(d) By applying Urysohn's lemma, show that every T_4–space is a Ty-chonoff space. 5

3. (a) Prove that a compact Hausdorff space is T_4–space. 5

Or

Show that intersection of closed and compact subset of a space is compact. Does the result is ture for intersection of two compact subsets ? 4+1=5

(b) Let (X, T) and (Y, T*) be two topological space such that X is compact and Y is Hausdorff. Let $f : X \rightarrow Y$ be a continuous function. Show that $\overline{f(A)} = f(\bar{A})$ for every $A \subseteq X$. 5

Or

Show that every real valued continuous function from a compact space is a quotient function. 5

(c) Let (X, T) be a topological space and $(\hat{X}, \hat{T})$ be its Alexandroff compactification. Show that $(\hat{X}, \hat{T})$ is Hausdorff if and only if (X, T) is locally compact Hausdorff. 5

(d) Let X be a compact Hausdorff space. Then for any $x \in X$, show that $Q(x) := \bigcap \{F \subseteq X \mid x \in F$ and F is clopen$\}$ is connected. 5

4. (a) For a collection of topological spaces, write a short note on the "comparison of the box and product topologies". 5

(b) Let (Y, T*) be a topological space and $\left(X = \prod_{\alpha} X_\alpha \; T\right)$ be a product space. Then show that a function $f : Y \rightarrow X$ is continuous if and only if for each projection $\pi_\alpha : X \rightarrow X_\alpha$, the composite mapping $\pi_\alpha \circ f : Y \rightarrow X_\alpha$ is continuous. 5

(c) Let $\{(X_\alpha, T_\alpha) \mid \alpha \in J\}$ be an arbitrary collection of normal spaces and T be the product topology on $X := \prod_{\alpha \in J} X_\alpha$.

Then for any be two disjoint closed subsets F_1 and F_2 of (X, T), can you construct a continuous function $f : X \rightarrow [0, 1]$ such that $f(F_1) = 1$ and $f(F_2) = 0$? Justify your answer. 1+4=5

(d) Let $\{(X_\alpha, T_\alpha) \mid \alpha \in J\}$ be an arbitrary collection of topological spaces and T be the product topology on $X := \prod_{\alpha \in J} X_\alpha$. Show that product space is compact if and only if each space is compact. 5

Or

Let $\left(X := \prod_{\alpha \in J} X_\alpha, T\right)$ be the product space of an indexed family of spaces $\{(X_\alpha, T_\alpha) \mid \alpha \in J\}$. Show that X is connected if and only if for each $\alpha \in J$, X_α has the corresponding property. 5

(Symbols are in their usual meaning.)