2017

MATHEMATICS

Paper : MTC 205

TOPOLOGY

Full Marks: 80

Time: 3 hours

The figures in the margin indicate full marks for the questions

1. (a) Let $(X, \mathscr{T})$ be a topological space. Show that a subcollection $\mathscr{B}$ of $\mathscr{T}$ is a base for $\mathscr{T}$ if and only if for each $G \in \mathscr{T}$ and for each $x \in G$ there is $B \in \mathscr{B}$ such that $x \in B \subseteq G$. 5

(b) Let A be a subset in a topological space $(X, \mathscr{T})$. Show that $A \cup A^d$ is closed in X. Hence show that $\bar{A} = A \cup A^d$. $3 + 2 = 5$

or

Let X be an infinite set and $f : \mathbb{P}(X) \to \mathbb{P}(X)$ be a function defined by

$$f(A) := \begin{cases} A, & \text{if } A \text{ is finite.} \\ X, & \text{if } A \text{ is infinite.} \end{cases}$$

Show that f satisfies the Kuratowski closure axioms. Hence find the topology induced by f. $3 + 2 = 5$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} x, & \text{if } x \leq 1. \\ x + 2, & \text{if } x > 1. \end{cases}$$

Check whether

i. f is $\mathscr{U} - \mathscr{U}$ is continuous.

ii. f is $\mathscr{T} - \mathscr{T}$ is continuous. 5

P.T.O.

(d) Let f be a continuous open function from a topological space $(X, \mathscr{T})$ onto another space $(Y, \mathscr{T}^*)$. Show that $\mathscr{T}^*$ is the quotient topology on Y relative to f. 5

2. (a) Show that in a second countable space every non-empty collection of disjoint open subsets of X is countable. 5

or

Show that In a second countable space every open set can be expressed as a countable union of open sets. 5

(b) Show that second countability implies the Lindelöf. Does the converse is true? $4 + 1 = 5$

(c) Show that every open subspace of a separable space is separable. Does the result is true for every subset? $4 + 1 = 5$

(d) Show that a topological space is T_1 if and only if every singleton subset is closed. 5

(e) Let X be a first countable space. Then show that X is T_2-space if every convergent sequence in X has a unique limit point. 5

3. (a) Show that a compact subset of a Hausdorff space is closed. Does the converse is true? $4 + 1 = 5$

(b) Show that a every compact Hausdorff space is Tychonoff space. 5

or

Show that every compact regular space is normal. 5

(c) Show that a dense subset of a locally compact Hausdorff space, is locally compact if and only if it is open. 5

or

Show that in a locally compact Hausdorff space, a subset is locally compact if and only if it is locally closed subset. 5

P.T.O.

(d) Show that a topological space is disconnected if and only if there exists a non-empty clopen proper subset of X. Hence show that $\mathbb{R}$ with lower topology is disconnected. $3 + 2 = 5$

or

Show that the components of a totally disconnected space X are singleton subset of X. 5

4. (a) Let $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in J\}$ be an arbitrary collection of topological spaces and $\mathscr{T}$ be a topology on $X := \prod_{\alpha \in J} X_\alpha$. Show that $\mathscr{T}$ is the product topology if and only if $\mathscr{T}$ is the smallest topology for which the projections are continuous. 5

(b) Let F_1 and F_2 be two disjoint closed subsets of $\mathbb{R}_{\mathscr{L}} \times \mathbb{R}_{\mathscr{L}}$. Can you find a continuous function $f : \mathbb{R}_{\mathscr{L}} \times \mathbb{R}_{\mathscr{L}} \to [0, 1]$ such that $f(F_1) = 1$ and $f(F_2) = 0$. Justify your answer. Where $\mathscr{L}$ is the lower limit topology on $\mathbb{R}$. $2 + 3 = 5$

(c) Let $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in J\}$ be an arbitrary collection of topological spaces and $\mathscr{T}$ be the product topology on $X := \prod_{\alpha \in J} X_\alpha$. Show that product space is compact if and only if each space is compact. 5

or

Let $(X := \prod_{\alpha \in J} X_\alpha, \mathscr{T})$ be the product space of an indexed family of spaces $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in J\}$. Show that X is connected if and only if for each $\alpha \in J$, X_α has the corresponding property. 5

(Symbols are in their usual meaning)

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