2017

MATHEMATICS

PAPER : MAT 301

MEASURE THEORY

FULL MARKS:80 Time :3 hours

{ The figures in the margin indicate full marks for the question.}

1. Answer any four of the following questions 5X4=20

(a) Prove that every countable set is L-measurable and its measure is zero.

(b) Let E_1 and E_2 be two measurable subsets of **R**, show that $m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2)$

(c) Prove that the outer measure of an interval is equal to its length.

(d) Prove that a set A is measurable if and only if a open set G containing A and a closed set H contained in A can be determined as $|G| - |H| < \in, \, for \in > 0$

(e) Prove that union of two measurable sets is also measurable.

2. Answer any four of the following questions 5X4=20

(a) Define measurable function. Define characteristic function and write their properties. Prove that characteristic function of a set A is measurable if and only if A is measurable .

(b) Let g be a measurable real valued function defined on a measurable set E and f be continuous real valued function defined on **R**. Then show that the composition fog is measurable function on E. Hence show that | g | is measurable on E.

(c) Give four equivalent definitions for the Lebesgue measurability of a real valued function and prove their equivalence.

Contd...2

(d) Let f and g be the measurable functions over a measurable space E. Prove that $f \cup g$ and $f \cap g$ are measurable functions. Give an example of a function for which $E(f=a)$ is measurable while the function f is not measurable.

(e) Show that every measurable function f defined on E can be expressible as difference of two non negative measurable functions on E.

3. Answer the following questions:(Any four) 5X4=20

(a) Let f be a integrable over E. Show that (i) $|f|$ is also integrable over E. (ii) For $\in > 0$, there is a subset E_0 of E with finite measure such that $\int_{E-E_0} |f| < \in$.

(b) Let f be a bounded function defined on [a, b]. Then prove that every upper sum is greater than or equal to every lower sum for f.

(c) Let f be a bounded function defined on [a, b] and if P be a measurable partition of [a, b], then $\text{Sup}_Q\, L[f; Q] \leq$" $\inf_P\, U[f; P]$, where supremum and infimum are taken over all measurable partitions Q and P of [a, b].

(d) Let f be a bounded function defined on [a, b] and f is R-integrable on [a, b] then prove that f is also L-integrable on [a, b] and $L\int_a^b f = R$. Cite an example to show that the converse is not true.

(e) Let f be a bounded function defined on [a, b]. Then prove that the function f is L-integrable, if and only if for each $\in > 0$, there exists a measurable partition P of [a, b] such that $U[f; P] - L[f; P] < \in$.

4. Answer the following questions:(Any four) 5X4=20

(a) Let f be a convex function on $(-\infty, \infty)$ and ϕ be integrable function on [0, 1], then prove that $f\left[\int_0^1 \phi(t)\,dt\right] \leq \int_0^1 f(\phi(t))\,dt$.

Contd...3

(b) If $f, g \in L^p$; $1 \leq$" $p < \infty$ then prove that $f + g \in L^p$ and $\|f + g\| \leq$" $\|f\|_p + \|g\|_p$.

(c) Prove that L^p space is complete.

(d) Let E be a measurable set, $1 \leq$" $p < \infty$, q be conjugate of p and g be an element of $L^p(E)$. Let T be functional on $L^p(E)$ defined by $T(f) = \int_E g.f$, for $f \in L^p(E)$. Show that T is bounded linear functional on $L^p(E)$ and $\|T\| = \|g\|_q$.

(e) Let E be a measurable set, $1 \leq$" $p < \infty$ and q be conjugate of p. Let $\{f_n\}$ converges weakly to f in $L^p(E)$ and $\{g_n\}$ converges strongly to g in $L^p(E)$. Show that

=. $\lim_{n\to\infty} \int_E g_n \cdot f_n = \int_E g \cdot f$.

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