2015

MATHEMATICS

Paper : 101

REAL ANALYSIS

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Attempt each of the following :

(a) Can it be possible to arrange the real numbers in a sequence? 1

(b) Does there exists a sequence of real valued continuous functions $\{f_n\}_{n=1}^{\infty}$ defined on $[-2, 2]$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on $[-2, 2]$, defined by

$$f(x) := \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq 0. \\ 1, & \text{if } x \neq 0. \end{cases}$$

Justify your answer. 1 + 1 = 2

(c) State the basic restriction on 'α' in the Reimann-Steiltjes integral $\int_a^b f d\alpha$. 1

P.T.O.

(d) What can you say about the completeness of a discrete metric space? 1

2. (a) Show that a countable union of countable sets is countable. Hence show that the set of irrational numbers is not countable. 3 + 2 = 5

(b) Show that the set of infinite binary sequence is not countable. 5

(c) Find the cardinality of $\mathbb{Z}[x]$, the set of all polynomial with integer co-efficient.

or

Find the cardinality of $\mathbb{Z}[i] := \{a + ib \mid a, b \in \mathbb{Z}\}$ 5

3. (a) Attempt any two of the following:

i. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on a closed interval $[a, b] \subseteq \mathbb{R}$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on [a, b]. Let $\forall n \in \mathbb{N}$, f_n is integrable on [a, b]. Show that

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f\,dt, \forall n \in \mathbb{N},$$ 5

ii. Prove or disprove that the sequence $< f_n >$, where

$$fn(x) := \frac{x}{1+nx^2}, \forall \chi \in \mathbb{R}$$

is uniformly convergent on anyclosed interval. 5

iii. State the Weierstrass Approximation theorem for uniform approximation of a realvalued continuous function defined on a closed interval. Applying this theorem, show that if $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 x^n f(x)\,dx = 0$, for n = 0,1, 2, ..., then $f(x) = 0; \forall x \in [0,1]$. 1 + 4 = 5

(b) Find the Fourier coecients and Fourier series of the function f defined by $f(x) := \begin{cases} 0, & \text{if } \pi \le x < 0. \\ 1, & \text{if } 0 \le x \le \pi. \end{cases}$ and $f(x+2\pi) = f(x)$

5

or

Explain in brief, how Fourier series enable us to synthesize the sounds of conventional musical instruments? 5

4. Attempt any two of the following:

(a) Show that the function

$f(x, y) := x^4 + x^2y + y^2$, $(x, y) \in \mathbb{R}$ has a minimum at (0, 0). 5

(b) Let the roots of the equation in λ, $(\lambda - x^3) + (\lambda - y^3) + (\lambda - z^3) = 0$ be u, v, w.

Show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = -2\dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$. 5

(c) Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$. 5

5. (a) Attempt any two of the following:

i. Show that a function of bounded variation can be ex pressible as a difference of two monotone increasing functions. Does the converse is true? 4 + 1 = 5

ii. Show that the function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } 0<x\le 1. \\ 0, & \text{if } x=0. \end{cases}$$

is of bounded variation on [0, 1]. 5

iii. Compute the positive, negative and the total variation functions of

$f(x) := 3x^2 - 2x^3$ for $-2 \le x \le 0$. 5

(b) Let f be a continuous real valued function defined on the interval [a, b] such that $f \in R(\alpha)$. Show that $\exists \xi \in [a, b]$ such that $\int_a^b f\, d\alpha = f(\xi)\{\alpha(b) - \alpha(a)\}$.

Can it be possible to find ξ in the open interval]a, b[?

4 + 1 = 5

or

Lef $f(x) := x$ and $\alpha(x) := x + [x]$, $x \in \mathbb{R}$, [x] stands for greatest integer function. Show that $\int_1^3 f d\alpha$ is exists and then evaluate $\int_1^3 f d\alpha$. 2 + 3 = 5

6. (a) Let f and g be two continuous functions from a metric space (X, d_1) into the metric space (Y, d_2) such that $f(x) = g(x)$, $\forall x \in A$, where A is a non empty subset of X. Show that

$f(x) = g(x)$, $\forall x \in \overline{A}$.

Hence show that if f and g are two real valued continu ous functions defined on $\mathbb{R}$ such that $f(x) = g(x)$, $\forall \chi \in \mathbb{Q}$, then $f(x) = g(x)$, $\forall x \in \mathbb{R}$. 4 + 1 = 5

or

Let f be a real-valued function defined on a metric space (X, d). Show that f is continuous on X iff for any $c \in \mathbb{R}$, the following sets

$\{x | f(x) < c\}$ and $\{x | f(x) > c\}$ are open in X. 5

(b) Let (X, d) be a compact metric space. Let $\mathcal{F} := \{F | F$ is a compact subset of X$\}$ and $\mathcal{O} := \{G | G$ is an nonempty proper d-open subset of X$\}$. Can you say that X is discon nected iff $\mathcal{F} \cap \mathcal{O} \neq \emptyset$. Justify your answer. 1 + 4 = 5

(c) Show that $D := \{(x, y) | x \neq 0, y = \sin \frac{1}{x}\}$ is a discon nected subset of $\mathbb{R}^2$. 5

or

Show that a metric space is connected iff every real-val ued continuous function 'f' has the intermediate value property. 5

(d) Show that $\mathbb{Q}$ is of rst category in $\mathbb{R}$ with respect to the usual metric. 5

or

Let (X, d) be a complete metric space such that each $x \in X$ is a limit point of X. By using Baire's category theorem,show that X is uncountable. 5

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