BU/EX/PG/SEM-1/101

2018

MATHEMATICS

MAT 101

REAL ANALYSIS

Full Marks: 80

Time: 3 hours.

The figures in the margin indicates full marks for the questions

1. Attempt *each* of the following:

   (a) Can it be possible to arrange the real numbers in a sequence? 1

   (b) Does there exists a sequence of real valued continuous functions $\{f_n\}_{n=1}^{\infty}$ defined on $[-2, 2]$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on $[-2, 2]$, defined by $f(x) := \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0. \\ 1, & \text{if } x = 0. \end{cases}$
   Justify your answer. $1 + 1 = 2$

   (c) State the basic restriction on 'α' in the Reimann-Steiltjes integral $\int_a^b f d\alpha$. 1

   (d) What can you say about the completeness of a discrete metric space? 1

2. (a) Show that a countable union of countable sets is countable. Hence show that the set of irrational numbers is not countable. $3 + 2 = 5$

   (b) Show that the set of infinite binary sequence is not countable. 5

   (c) Find the cardinality of $\mathbb{Z}[x]$, the set of all polynomial with integer co-efficient.

   or

   Find the cardinality of $\mathbb{Z}[i] := \{a + ib \mid a, b \in \mathbb{Z}\}$. 5

3. (a) Attempt *any two* of the following:

   i. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions defined on a closed interval $[a, b] \subseteq \mathbb{R}$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f on $[a, b]$. Let

$\forall\, n \in \mathbb{N}$, f_n is integrable on $[a, b]$. Show that $\int_a^x f dt = \lim_{n\to\infty} \int_a^x f dt$, $\forall\, n \in \mathbb{N}$. 5

ii. Prove or disprove that the sequence $< f_n >$, where

$$f_n(x) := \frac{x}{1+nx^2}, \ \forall\, x \in \mathbb{R}$$

is uniformly convergent on any closed interval. 5

iii. State the Weierstrass Approximation theorem for uniform approximation of a real valued continuous function defined on a closed interval. Applying this theorem, show that if $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $\int_0^1 x^n f(x)dx = 0$, for $n = 0, 1, 2, \ldots$, then $f(x) = 0$, $\forall\, x \in [0, 1]$. $1 + 4 = 5$

(b) Find the Fourier coefficients and Fourier series of the function f defined by

$$f(x) := \begin{cases} 0, & \text{if } -\pi \le x < 0. \\ 1, & \text{if } 0 \le x \le \pi. \end{cases} \text{ and } f(x + 2\pi) = f(x).$$ 5

or

Explain in brief, how Fourier series enable us to synthesize the sounds of conventional musical instruments? 5

4. Attempt *any two* of the following:

(a) Show that the function

$f(x, y) := x^4 + x^2y + y^2$, $(x, y) \in \mathbb{R}$ has a minimum at $(0, 0)$. 5

(b) Let the roots of the equation in λ, $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ be u, v, w. Show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$. 5

(c) Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$. 5

5. (a) Attempt *any two* of the following:

i. Show that a function of bounded variation can be expressible as a difference of two monotone increasing functions. Does the converse is true? $4 + 1 = 5$

ii. Show that the function $f : [0, 1] \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } 0 < x \le 1. \\ 0, & \text{if } x = 0. \end{cases}$$

is of bounded variation on $[0, 1]$. 5

P.T.O.

iii. Compute the positive, negative and the total variation functions of $f(x) := 3x^2 - 2x^3$ for $-2 \le x \le 0$. 5

(b) Let f be a continuous real valued function defined on the interval $[a, b]$ such that $f \in \mathscr{R}(\alpha)$. Show that $\exists\, \xi \in [a, b]$ such that $\int_a^b f\, d\alpha = f(\xi)\{\alpha(b) - \alpha(a)\}$. Can it be possible to find ξ in the open interval $]a, b[$? $4 + 1 = 5$

or

Let $f(x) := x$ and $\alpha(x) := x + \lfloor x \rfloor$, $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for greatest integer function. Show that $\int_1^3 f d\alpha$ is exists and then evaluate $\int_1^3 f d\alpha$. $2 + 3 = 5$

6. (a) Let f and g be two continuous functions from a metric space (X, d_1) into the metric space (Y, d_2) such that $f(x) = g(x)$, $\forall\, x \in A$, where A is a non empty subset of X. Show that $f(x) = g(x)$, $\forall\, x \in \overline{A}$.

Hence show that if f and g are two real valued continuous functions defined on $\mathbb{R}$ such that $f(x) = g(x)$, $\forall\, x \in \mathbb{Q}$, then $f(x) = g(x)$, $\forall\, x \in \mathbb{R}$. $4 + 1 = 5$

or

Let f be a real-valued function defined on a metric space (X, d). Show that f is continuous on X iff for any $c \in \mathbb{R}$, the following sets

$$\{x \mid f(x) < c\} \text{ and } \{x \mid f(x) > c\} \text{ are open in } X.$$ 5

(b) Let $f : \mathbb{R} \mapsto \mathbb{R}$, be a function. Show that the set of all points at which f is continuous points is G_δ set. 5

(c) Show that $D := \{(x, y) \mid x \ne 0, y = \sin\frac{1}{x}\}$ is a disconnected subset of $\mathbb{R}^2$. 5

or

Show that a metric space is connected iff every real-valued continuous function 'f' has the intermediate value property. 5

(d) Show that $\mathbb{Q}$ is of first category in $\mathbb{R}$ with respect to the usual metric. 5

or

Let (X, d) be a complete metric space such that each $x \in X$ is a limit point of X. By using Baire's category theorem, show that X is uncountable. 5

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