2018

MATHEMATICS

MAT 201

FUNCTIONAL ANALYSIS

Full Marks: 80

Time: 3 Hours.

1. Answer any ***four*** questions: **5X4**

(a) Prove that- L_p space is a normed linear space under the norm

$$\|f\|_p = \left[\int_X |f(x)|^p d\mu(x)\right]^{1/p}$$

(b) Let M be a closed linear subspace in a normed space N. For each coset $x + M$ in the quotient space, (N/M)

$$\|x + M\| = inf\{\|x + m\|: m \in M\}$$

Then show that N/M is complete.

(c) Define bounded linear functional and describe in L_p space

(d) If a normed space X has the property that the closed unit ball,

$M = \{x \in X: \|x\| \leq 1\}$ is compact then show that X is finite dimensional.

(e) Prove that- in a finite dimensional Normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

 P.T.O.

2. Answer any *two* questions: 6X2

(a) Let $T: X \to Y$ be a linear mapping where X and Y be normed linear spaces. Then show that the following statements are equivalent to one another

(i) T is continuous at any point x_o,

(ii) T is bounded,

(iii) If $S = \{x : \|x\| \leq 1\}$ is closed unit sphere in X then its image is a bounded set in Y.

(b) Let f be a bounded linear functional on a subspace Z of a normed space X. Then show that there exists a bounded linear functional f^* on X which is a extension of f to X and has the same norm, $\|f^*\|_X = \|f\|_Z$.

(c) Define extension mapping. Let X be a normed space and let $x_o(\neq 0) \in X$. Then show that there exists a bounded linear functional $f^\wedge$ on X such that $\|f^\wedge\| = 1$, $f^\wedge(x_o) = \|x_o\|$.

3. Let X be a complex vector space and p be a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace z of X and satisfies $|f(x)| \leq p(x)\ \forall x \in Z$. Then show that f has a linear extension $f^\wedge$ from Z to X, satisfying $|f^\wedge(x)| \leq p(x), \forall x \in X$. 8

4. Answer any *two* questions: 8X2

(a) State and prove open mapping theorem.

(b) Let $T: D(T) \to Y$ be a closed linear operator, where X and Y be Banach spaces and $D(T) \subset X$. If $D(T)$ is closed in X then show that T is bounded.

(c) If Y is a Banach space then show that $B(X, Y)$ is a Banach space.

5. Answer any ***three*** questions: 4X3

(a) If I is the inner product space then show that $\sqrt{\langle x, x\rangle}$ has the property of a norm.

(b) If $x, y \in H$, Hilbert space then show that $|\langle x, y\rangle| \leq \|x\|\|y\|$.

(c) Let H_1 and H_2 be Hilbert spaces, $S: H_1 \to H_2$ and $T: H_1 \to H_2$ be bounded linear operators and α any scalar. Then show that

(i) $(\alpha T)^* = \bar{\alpha} T^*$

(ii) $(T^*)^* = T$

(d) If P is a projection on a Hilbert space H. Then prove that

(i) $P \geq 0$

(ii) $0 \leq P \leq I$

(iii) $\|Px\| \leq \|x\|,\ \forall x \in H$.

6. Answer any *two* questions: 6X2

(a) Let (e_k) be an orthonormal sequence in an inner product space X. Then show that $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$. (Bessel's Inequality)

(b) Prove that- Every Bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, $f(x) = \langle x, z \rangle$ where z depends on f, is uniquely determined by f and has the norm $\|z\| = \|f\|$.

(c) If P is a projection on a Hilbert space H with range M and null space N then show that $M \perp N$ if and only if P is self adjoint.

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