2018

MATHEMATICS

MAT 301

MEASURE THEORY

Full Marks: 80

Time: 3 hours.

The figures in the margin indicate full marks for the questions

1. Answer any ***four*** questions: **5X4**

(a) For any sequence of sets $\{E_i\}$ show that

$$m^*(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i) .$$

(b) Prove that- The class M is a σ algebra.

(c) Show that- any set A is measurable if and only if an open set $A \subseteq G$ and a closed set $H \subseteq A$ such that $|G| - |H| <\in$.

(d) Establish the existence of a non measurable set.

(e) Show that- A is measurable is if and only if A^c is measurable.

2. (a) Answer any ***two*** questions: **6X2**

(i) If $f(x) = K\ a.e.$, is a constant function on a measurable set E then show that $\int_a^b f(x)dx = K.m(E)$.

P.T.O.

(ii) Verify bounded convergence theorem for the sequence of functions

$$f_n(x) = \frac{1}{\left(1+\frac{x}{n}\right)^n}, x \in [0,1], n \in N.$$

(iii) Let $f(x) = 1$, if x is an irrational number in $[a,b]$ and $f(x) = 2$, if x is a rational number in $[a,b]$. Evaluate $\int_a^b f(x)dx$.

(b) Answer any *two* questions: **4X2**

(i) Show that the set $E \subseteq [a,b]$ and its characteristic function φ_E are both measurable and non measurable.

(ii) If $f = g$ a.e.and f is measurable function then show that g is also measurable function.

(iii) Show that the function f on $[a,b]$ is measurable if and only if any one of the following conditions hold: $\{x: f(x) > \alpha\}$; $\{x: f(x) \geq \alpha\}$; $\{x: f(x) < \alpha\}$ and $\{x: f(x) \leq \alpha\}$ are measurable sets for every real α respectively.

3. (a) Answer any *two* questions: **8X2**

(i) Let $\mu(X) < \infty$, $\{f_n\}$ is uniformly integrable, $f_n \to f$ $a.e.$, and $|f(x)| < \infty$ $a.e.$, then show that $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.

(ii) Let A be any measurable subset of $[a,b]$ with finite measure. Let $\{f_n\}$ be a sequence of measurable functions such that for $x \in A$

$$0 \leq f_1(x) \leq \cdots \leq f_n(x) \leq \cdots$$

If $\lim_{n\to\infty} f_n(x) = f(x)$, then show that f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

(iii) State and prove- Fatou's Lemma. Deduce Lebesgue monotone convergence theorem from it.

(b) If f is Riemann integrable and bounded over the finite interval $[a,b]$, then show that f is integrable and $R\int_a^b f(x)dx = \int_a^b f(x)dx$. **4**

4. (a) Answer any *two* questions: **8 x 2**

(i) Show that- Every Cauchy sequence $\{f_n\}$ in the L^p space converges to a function in L^p space.

(ii) If f is a bounded measurable function defined on $[a,b]$, then for given $\epsilon > 0$, show that there exists a continuous function g on $[a,b]$, such that

$$\|f - g\|_2 < \epsilon.$$

(iii) Let $[\![X, S, \mu]\!]$ be a σ-finite measure space and let G be a bounded linear functional on $L^1(X,\mu)$. Then show that there exists a unique $g \in L^\infty(X,\mu)$ such that $G(f) = \int fg\, d\mu$ for each $f \in L^1(\mu)$.
Also $\|G\| = \|g\|_\infty$.

(b) Let $p \geq 1$ and let $f, g \in L^p(\mu)$ then show that **4**

$$\left(\int |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}}$$

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