2018
MATHEMATICS
MAT 403
ELECTIVE III (NUMBER THERORY II)

Full Marks: 80

Time: 3 hours

The figures in the margin indicate full marks for the questions

1. (a) For $n \geq 1$, show that $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$. 5

or

For any two positive integers m, n, show that $\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)}$, where $d := (m, n)$.

(b) Show that a multiplicative function f is completely multiplicative if and only if $f^{-1}(n) = \mu(n)f(n), \ \forall\, n \geq 1$. 5

or

For any multiplicative function f, show that $\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p))$, where p is a prime number. 5

(c) Show that the average order of $\varphi(n)$ is $3n/\pi^2$. 5

or

Show that the set of lattice points visible from the origin has density $6/\pi^2$. 5

(d) For all $x \geq 1$, show that $|\sum_{n \leq x} \frac{\mu(n)}{n}| \leq 1$. Under what condition the equality will hold. 4+1= 5

(e) For $n \geq 1$, show that the n^{th} prime p_n satisfies the inequalities $\frac{1}{6} n \log n < p_n < 12(n \log n + n \log \frac{12}{e})$. 5

Contd...2

2. (a) Show that there is exactly $\frac{p-1}{2}$ quadratic residue and $\frac{p-1}{2}$ quadratic residue mod p. 5

or

Show that if $(p, a) = 1$, then $x^2 \equiv a(\mod p)$ has two solutions. 5

(b) If p is an odd prime and a, b be two integers such that $(a, p) = 1$ and $(b, p) = 1$. Then show either *one* of the following 5

i. $(\frac{a}{p}) \equiv a^{\frac{p-1}{2}}(\mod p)$.

ii. $a \equiv b(\mod p) \Rightarrow (\frac{a}{p}) = (\frac{b}{p})$.

(c) If $P(m, n)$ be a lattice point in the first quadrant and $(m\,n) = 1$, then show that there does not exists any lattice point on the line OP, excluding the points O and P. 5

(d) Show that the congruence $x^2 \equiv 15(\mod 1093)$ has no solution. 5

3. For $s > 1$, show *any three* of the following $5 \times 3 = 15$

(a) $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$.

(b) $\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$.

(c) $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.

(d) $\frac{\sigma_{s-1}(m)}{m^{s-1}\zeta(s)} = \sum_{n=1}^{\infty} \frac{c_n(m)}{n^s}$.

4. (a) State and prove Euler's pentagonal-number theorem. 2+8=10

(b) For each nonnegative integer n, show *any one* of the following 10

i. $p(5m + 4) \equiv 0(\mod 5)$.

ii. $p(11m + 6) \equiv 0(\mod 11)$

(Symbols are in their usual meaning.)

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