2016

MATHEMATICS

MAT 201
FUNCTIONAL ANALYSIS

Full Mark : 80

Time : 3 Hrs

Figures in the right hand margin indicate full marks for the question

1. Answer any four questions: 5X4=20

(a) Prove that every normed linear space is metric space but every metric space may not be a norm linear space.

(b) Prove that the linear space C^2 is a Banach space with the norm $\| x \| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$

(c) What is equivalent norm? Prove that on a finite dimensional normed linear space any two norms are equivalent.

(d) Let Y be a proper closed subspace of a normed linear space X. Then prove that for every $\theta \in (0, 1)$ there exists $z \in X$ such that $\| z \| = 1$ and $\| z-y \| \geq \theta$ for $y \in Y$.

P.T.O.

(e) Prove that on finite dimensional normed linear space every linear operator is bounded.

2. Answer any two questions 7X2=14

(a) State and prove open mapping theorem.

(b) Let N and N' be two normed linear spaces and D be sub-space of N. Then show that a linear transformation $T: D \rightarrow N'$ is closed if and only if its graph T_G is closed.

(c) Show that the mapping $J: N' \rightarrow N^{**}$ such that $J(x)=F_x$ for all x in N defines an isometric isomorphism on N into N^{**}, where F_x is function on N^* defined by $F_x(f)=f(x)$, for $f \in N^*$.

3. Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ be a non-empty set of bounded linear transformations from B into N having the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then show that $\{\|T_i\|\}$ is a bounded set of numbers. 6

OR

Let B and B' be two Banach spaces. If T is a continuous linear transformation of B onto B', then show that T is an open mapping.

4. Answer any four questions 5X4=20

(a) If x and y be any two vectors in a Hilbert space H, then prove that

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4<x, y>.$$

(b) Prove that the space C[a, b] is not Hilbert space.

(c) Show that $T: l^\infty \rightarrow l^\infty$ defined by $Tx=(x_1, x_2/2, x_3/3, \dots\dots\dots)$ for some $x=(x_1, x_2, x_3, \dots\dots)$ is linear and bounded.

(d) Let X be an inner product space and $\{u_1, u_2, \dots\dots u_n \dots\dots\}$ be a countable orthonormal set in X and $k_1, k_2 \dots\dots$ belongs to scalar field K. If $\sum k_n U_n$ converges to some $x \in X$ then show that $<x, u_n> = k_n$ for each n and $\sum_{n=1}^{\infty} |kn|^2 \leq \infty$.

(e) Let $x_1(t)=t^2$, $x_2(t)=t$, and $x_3(t)=1$. Orthonormalize x_1, x_2, x_3 in order on [-1, 1] with respect to inner product $<x, y> = = \int_{-1}^{1} x(t)y(t)dt$.

5. Answer any four questions 5X4=20

(a) Prove that for any bounded linear operator T on a Hilbert space H, the operators $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2}$ are self adjoint.

(b) Let A and B $\supseteq$ A be nonempty subsets of an inner product space X, then show that

(i) $A^{\perp\perp} \supseteq A$

(ii) $A^{\perp} \supseteq B^{\perp}$

(c) Show that a subspace Y of a Hilbert space H is closed in H if and only if $Y = Y^{\perp}$

(d) If T be arbitrary operator on a Hilbert space H and a, b be the scalars such that $|a| = |b|$, then prove that $aT + BT^{*}$ is a normal operator.

(e) Let X be inner product space and Y be a complete subspace of X. Then prove that for any $x \in X$, there exists a unique $y \in Y$ such that $\| x-y \| = d(x, y)$, moreover z=s-y is in $Y^{\perp}$.

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