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63/2 (SEM-4) MAT 402 (A,B)

2022

MATHEMATICS

(Theory Paper)

Paper Code : MAT-402 (A)

(Advanced Functional Analysis)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct option : $1\times6=6$

(a) Which of the following is not a property of weak topology ?

(i) Let X be a topological vector space (with topology τ) whose dual X^* separates points on X.

(ii) The X^* topology is called weak topology, τ_w of X.

(iii) $\tau \subset \tau_w$

(iv) X_w is locally convex whose dual is also X^*

(b) Find the incorrect statement :

(i) In C* algebra if p is non-zero projection then $\|p\| \neq 1$.

(ii) In C* algebra if u is unitary then $\|u\| = 1$

(iii) A C* algebra is a Banch * –algebra, A such that $\|aa^*\| = \|a\|^2$, $a \in A$.

(iv) C* algebra with indentity element, e : $\|e\| = 1$

(c) Let A be a C* algebra then which of the following is true :

(i) The set A^+ is equal to $a^*a : a \in A$.

(ii) If $0 \leq a \leq b$ then $\|a\| \leq \|b\|$

(iii) If a, b positive invertible elements then $a \leq b \Rightarrow 0 \leq b^{-1} \leq a^{-1}$

(iv) All of the above.

(d) Let $X \neq 0$ be a complex normed space and $T : D(T) \rightarrow X$ a linear operator with domain $D(T) \subset X$. Which of the following is the false statement ?

(i) $R_\lambda(T)$ exists

(ii) $R_\lambda(T)$ bounded

(iii) $R_\lambda(T)$ is defined on a set which is not dense in X.

(iv) The complement $\sigma(T) = C - \rho(T)$ is spectrum of T, where $\rho(T)$ is resolvent of T.

(e) Let $T: H \rightarrow H$ be a positive, bounded, self-adjoint operator a complex Hibert space H, and $\leqq$ is a partial ordering. Then find the incorrect statement :

(i) $\langle Tx, x \rangle$ is not real

(ii) $T_1 \leqq T_2$ if and only if $\langle T_1x, x \rangle \leqq \langle T_2x, x \rangle$

(iii) T is positive i.e., $0 \leqq T$ if and only if $0 \leqq \langle Tx, x \rangle$

(iv) $T_1 \leqq T_2$ if and only if $0 \leqq T_2 - T_1$

(f) Which of the following is correct ?

(a) Contration $\rightarrow$ contractive $\rightarrow$ non – expansive $\rightarrow$ Lipschitz continuous

(b) contractive $\rightarrow$ contration $\rightarrow$ non – expansive $\rightarrow$ Lipschitz continuous

(c) Lipschitz continuous $\rightarrow$ contration $\rightarrow$ contractive $\rightarrow$ non – expansive

(d) non – expansive $\rightarrow$ Lipschitz continuous $\rightarrow$ contration $\rightarrow$ contractive.

2. Answer the following questions : 2×5=10

(a) Define topological vector spaces with example.

(b) Distinguish betgween Banch Algebra and C*-Algebra.

(c) Define spectrum and resolvent set.

(d) Define contraction mapping and give an example.

(e) Let E be a set on a topological space X. Then prove that the following properties are equivalent :

(i) E is bounded.

(ii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in E and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of scalars in E such that $x_n \rightarrow 0$ as $n \rightarrow \infty$ then $a_n x_n \rightarrow 0$ as $n \rightarrow \infty$.

3. Answer any *six* of the following questions : 5×6=30

(a) Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some $x \in X$. Then prove that each of the following four properties implies other three :

(i) Λ is continuous

(ii) The null space $N(\Lambda)$ is closed

(iii) $N(\Lambda)$ is not dense in X.

(iv) Λ is bounded in some neighbourhood of V of 0.

(b) Prove that : "Every locally compact topological vector space X has finite dimension".

(c) Let (H_1, ϕ_1) and (H_2, ϕ_2) be two representations of a C*-algebra A with cyclic vectors x_1 and x_2 respectively. Then show that there exists a unitary operator $u : H_1 \to H_2$ such that $x_2 = u(x_1)$ and $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ if and only if $<\phi_1(a)(x_1), x_1> = <\phi_2(a)(x_2), x_2>$ for all $a \in A$.

(d) Show that : All matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values.

(e) If A is a banch algebra in which every non-zero element is invertible then prove that A is the complex field, $\mathbb{C}$.

(f) Show that : A bounded linear operator $P : H \to H$ on a Hilbert spece H is a projection if and only if $P^* = P = P^2$.

(g) Let A and B be two subsets of a topological vector space X where A be compact, B be closed and $A \cap B = \phi$. Then prove that 0 has a neighbourhood V such that $(A+V) \cap (B+V) = \phi$.

(h) If two bounded self-adjoint operator S and T on a Hilbert space H are positive and commute each other. Then show that their product, ST is positive.

(i) State and prove Banach fixed Point Theorem.

4. Answer any *two* of the following questions : 10×2=20

(a) State and prove Picards's Existence and Uniqueness Theorem. 2+8

(b) Let A be a unital Banach algebra generated by 1 and an element a. Then prove that A is abelian and the map $\hat{a} : \Omega(A) \to \sigma(a)$, $\tau \to \tau(a)$ is homomorphism. Validate this result for $A = l^1(Z)$, the set of all complex-valued function f on Z such that $\|f\|_1 = \sum_{n=\infty}^{\infty} |f(n)|$ is finite, and for any $f, g \in l^1(Z)$ their convolution is $f * g : z \to \mathbb{C}$ such that $(f*g)(m) = \sum_{n=-\infty}^{\infty} f(m-n)g(n)$. 2+8

(f) Which of the following is correct ?

(a) Contration $\rightarrow$ contractive $\rightarrow$ non – expansive $\rightarrow$ Lipschitz continuous

(b) contractive $\rightarrow$ contration $\rightarrow$ non – expansive $\rightarrow$ Lipschitz continuous

(c) Lipschitz continuous $\rightarrow$ contration $\rightarrow$ contractive $\rightarrow$ non – expansive

(d) non – expansive $\rightarrow$ Lipschitz continuous $\rightarrow$ contration $\rightarrow$ contractive.

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(e) Let E be a set on a topological space X. Then prove that the following properties are equivalent :

(i) E is bounded.

(ii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in E and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of scalars in E such that $x_n \rightarrow 0$ as $n \rightarrow \infty$ then $a_n x_n \rightarrow 0$ as $n \rightarrow \infty$.

3. Answer any *six* of the following questions : 5×6=30

(a) Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some $x \in X$. Then prove that each of the following four properties implies other three :

(i) Λ is continuous

(ii) The null space $N(\Lambda)$ is closed

(iii) $N(\Lambda)$ is not dense in X.

(iv) Λ is bounded in some neighbourhood of V of 0.

(b) Prove that : "Every locally compact topological vector space X has finite dimension".

(iii) Let $T: H \to H$ be a bounded self-adjoint operator on a complex Hilbert Space H. Then show that $\lambda \in \rho(t)$ if and only if $\|T_\lambda x\| \geq c\|x\|$; $c>0$, $x \in H$. $[T_\lambda = T - \lambda I]$ 10

5. Answer any *one* of the following questions : $14\times1=14$

(i) If φ is a linear functional on a Banach Algebra A, such that $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible $x \in A$, then show that $\varphi(x,y) = \varphi(x)\varphi(y)$ where $x, y \in A$. 14

(ii) Consider a Volterra integral equation : $u(x) = f(x) + \lambda\int_a^y K(x,t)u(t)\,dt$. Let f is continuous on [a,b] and the kernel k is continuous on the triangular region R is the xt-plane given by $a \leq t \leq y$ and $a \leq x \leq b$. Then prove that the Volterra integral equation has a unique solution u on [a, b] for every μ. Validate the result for $f(x) = x + \frac{1}{5x} - \frac{1}{5}$; $K(x,t) = \frac{1}{xt}$ and $\lambda = \frac{1}{5}$ where, $x \in [1,2]$. 10+4

(Theory Paper)

Paper Code : MAT-402 (B)

(Dynamical Systems)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions :

(a) Define dynamical systems. Give four examples of dynamical systems. 1=4=5

(b) Define orbit. If $F(x) = x^2 + 1$, find the first five points on the orbit of 0. 1+4=5

(c) Find an explicit formula for $D^2(x)$ and $D^3(x)$. Also write down a general formula of $D^n(x)$. 10

Or

Define tent map. Find an explicit formula for $T^2(x)$ and $T^3(x)$. Sketch the graphs.

2. Answer the following questions :

(a) Give graphical analysis for

(i) $F(x)=\frac{1}{3x}$

(ii) $F(x) = -2x+1$. 2½+2½=5

(b) Discuss the behavior of the resulting orbits under iterations of d for

(i) $x_0 = 0.3$

(ii) $x_0 = \frac{1}{14}$ 2½+2½=5

(c) Write in details about orbit analysis and phase portrait. 10

Or

Consider $F(x) = x^2 - 1.1$. Find the fixed points of F. Then use the fact that these points are also solutions of $F^2(x) = x$ to find the cycle of prime period 2 for F.

3. Answer the following questions :

(a) What is stability of a fixed point. How to use T and D for stability analysis of the fixed point (x^*, y^*). 5

(b) Define saddle point, saddle node, unstable node, stable focus and center. 5

(c) Construct and discuss a phase diagram for the pendulum equation $\ddot{x} + \alpha \sin x = 0$. 10

Or

Write in details about Duffing Oscillator. 10

4. Answer the following questions :

(a) Define Saddle-Node Bifurcation. 5

(b) Write about the Koch Snowflake with diagram. 5

(c) Identify the bifurcation for $G_\mu(x) = \mu x + x^3$, $\mu = -1, 1$. 10

Or

Define filled Julia Set and Julia Set. Describe the filled Julia Set for $Q_0(z) = z^2$.