2023

MATHEMATICS

Paper : MATHC3066

(Group Theory—I)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer from the following (any *six*) : 1×6=6

(a) The order of the symmetric group S_7 is

(i) 720

(ii) 5040

(iii) 120

(iv) 1440

(b) The inverse of the element 7 in the group $\mathbb{Z}_{12}$ is

(i) 5

(ii) 6

(iii) 7

(iv) 12

(c) If n is a prime number, then the set $\{1, 2, 3, ..., n\}$ is a group under the

(i) composition of addition of integers

(ii) composition of multiplication of integers

(iii) composition of multiplication modulo n

(iv) composition of addition modulo n

(d) The permutation (1243)(3521) is

(i) even

(ii) odd

(iii) neither even nor odd

(iv) both even and odd

(e) Let G be a group and $a \in G$ such that $|a| = n$. If $a^k = e$, where e is the identity element of G, then

(i) n divides $k-1$

(ii) k divides n

(iii) n does not divide k

(iv) n divides k

(f) Let G be the group such that $G = Z(G)$. Then

(i) G is not an abelian group

(ii) G is not a finite group

(iii) G is an abelian group

(iv) G is a finite group

(g) In the cyclic group $(\mathbb{Z}, +)$

(i) 1 is a generator but not −1

(ii) −1 is a generator but not 1

(iii) both 1 and −1 are generators

(iv) neither 1 nor −1 is a generator

(h) The number of elements in the dihedral group D_6 is

(i) 10

(ii) 11

(iii) 12

(iv) 13

(i) The identity element of a factor group $\frac{G}{H}$ is

(i) H

(ii) G

(iii) $\frac{G}{H}$

(iv) same as the identity element of G

(j) If N is a normal subgroup of a finite group G, then

(i) $\left|\frac{G}{N}\right| > \frac{|G|}{|N|}$

(ii) $\left|\frac{G}{N}\right| < \frac{|G|}{|N|}$

(iii) $\left|\frac{G}{N}\right| = \frac{|G|}{|N|}$

(iv) $\left|\frac{G}{N}\right| \neq \frac{|G|}{|N|}$

2. Answer any *five* of the following questions : 2×5=10

(a) Give an example of a group with 105 elements.

(b) Find the order of the permutation (1235)(24567).

(c) In a group, prove that the right cancellation law holds.

(d) Let $G = \{2^m : m \in \mathbb{Z}\}$ and $\varphi : \mathbb{Z} \to G$ be defined by $\varphi(n) = 2^n$, $n \in \mathbb{Z}$. Check whether φ is a homomorphism.

(e) If $\alpha = (21)(45)$ is an element of S_5, then find α^2.

(f) Examine if the set of all real numbers is a group under the composition of multiplication of real numbers.

(g) Compute $\mathbb{Z}_5 \oplus \mathbb{Z}_8$.

3. Answer any *six* of the following questions : 5×6=30

(a) Prove that the disjoint cycles of a permutation commute.

(b) Prove that the set of all 2×2 matrices with determinant 1 and having entries from the set of all rational numbers is a non-abelian group under matrix multiplication.

(c) Let H be a non-empty finite subset of a group G. If H is closed under the operation of G, then prove that H is a subgroup of G.

(d) Let G be a group and $a \in G$. Prove that the centralizer of a is a subgroup of G.

(e) Let f be a function from the group $(\mathbb{R}^+, \times)$ to the group $(\mathbb{R}, +)$ defined by $f(x) = \log_{10} x$, $x \in \mathbb{R}^+$. Show that f is an isomorphism. (Here $\mathbb{R}^+$ is the set of all positive reals)

(f) Let H be a subgroup of a group G. Prove that any two right cosets of H in G are either disjoint or identical.

(g) Suppose G is an abelian group with an odd number of elements. Show that the product of all elements of G is the identity element.

(h) Show that the set $G = \{1, 3\}$ is a group under multiplication modulo 4.

(i) Find all the distinct left cosets of $H = \{1, 11\}$ in $U(30)$.

(j) Prove that a subgroup N of a group G is a normal subgroup of G if and only if $xNx^{-1} = N$ for all $x \in G$.

4. Answer any *two* of the following questions : 10×2=20

(a) Let G be a finite abelian group and p be a prime that divides the order of G. Prove that G has an element of order p. 10

(b) *(i)* Find the centre of the symmetric group S_3. 7

(ii) The set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. Find the identity element. 3

(c) *(i)* Show that a subgroup of index 2 in a group G is a normal subgroup of G. 5

(ii) If a and b are two elements of a group G, then show that $(ab)^{-1} = b^{-1}a^{-1}$. 5

(d) State and prove Cayley's theorem. 1+9=10

5. Answer any *one* of the following questions : 14

(a) *(i)* Let H and K be two subgroups of a group G, where H is normal in G. Prove that

$$\frac{HK}{H} \cong \frac{K}{H \cap K}$$ 7

(ii) Let a be an element of order n in a group and let k be a positive integer. Prove that $\langle a^k \rangle = \langle a^{\gcd(n, k)} \rangle$ and

$$|a^k| = \frac{n}{\gcd(n, k)}$$ 4+3=7

(b) (i) Let $f : G \to G'$ be a homomorphism. Let $a \in G$ be such that $|a| = n$ and $|f(a)| = m$. Show that $|f(a)|$ divides $|a|$. Also, show that f is one-one if and only if $m = n$. 3+5=8

(ii) Let e be the identity of a group G. If $x \in G$ is such that $x^2 \neq e$ and $x^6 = e$, then prove that $x^4 \neq e$ and $x^5 \neq e$. What can you say about the order of x? 5+1=6

(c) (i) Let G be a finite group and $a \in G$. Prove that $a^{|G|} = e$, where e is the identity element of G. 5

(ii) Let $f : G \to G'$ be an isomorphism and $a \in G$. Prove that $G = \langle a \rangle$ if and only if $G' = \langle f(a) \rangle$. 5

(iii) Let G be an abelian group. Let $H = \{x \in G \mid x^2 = e\}$. Then prove that H is a subgroup of G. 4

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