63/1 (SEM–3) CC7/MATHC3076

2023

MATHEMATICS

Paper : MATHC3076

(PDE and Systems of ODE)

Full Marks : 60

Pass Marks : 24

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer (any *five*) : 1×5=5

(a) The general form of a first-order quasi-linear partial differential equation is

(i) $a(x, y)u_x + b(x, y)u_y = c(x, y, u)$

(ii) $a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$

(iii) $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$

(iv) $a(x, y)u_x^2 + b(x, y)u_y^2 = c(x, y, u)$

(b) A partial differential equation can be formed by

(i) eliminating arbitrary constant only

(ii) eliminating arbitrary constant and arbitrary function

(iii) eliminating arbitrary function only

(iv) integrating the given family of curves

(c) Which of the following represents canonical form of a parabolic equation?

(i) $u_{\eta\eta} = 0$

(ii) $u_{\xi\eta} = 0$

(iii) $u_{\alpha\alpha} + u_{\beta\beta} = 0$

(iv) $u_{\xi\eta} + u_{\eta\eta} = 0$

(d) The equation $u_t - c^2(u_{xx} + u_{yy}) = 0$, where $c \neq 0$, is known as

(i) wave equation

(ii) Laplace's equation

(iii) heat equation

(iv) Poisson's equation

(e) If the system of two linear differential equations in two unknown functions

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + F_1(t)$$
$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + F_2(t)$$

is to be homogeneous, then

(i) $F_1(t) < F_2(t), \forall\ t$

(ii) $F_1(t) = F_2(t) = k, k \neq 0, \forall\ t$

(iii) $F_1(t) > F_2(t), \forall\ t$

(iv) $F_1(t) = F_2(t) = 0, \forall\ t$

(f) Which of the following is a non-homogeneous wave equation?

(i) $u_{tt} = c^2 u_{xx},\ c \neq 0$

(ii) $u_{tt} = c^2 u_{xx} + h(x,t),\ c \neq 0$

(iii) $u_{tt} + u_{xx} = 0$

(iv) $u_t = c^2 u_{xx}, c \neq 0$

(g) Which of the following does not come under partial differential equations?

(i) Laplace's equation

(ii) One-dimensional wave equation

(iii) Heat equation

(iv) Equations of motion

(h) Two solutions

$$\begin{matrix} x = f_1(t) \\ y = g_1(t) \end{matrix} \quad \text{and} \quad \begin{matrix} x = f_2(t) \\ y = g_2(t) \end{matrix}$$

of a homogeneous linear system of two differential equations in two unknown functions x and y are linearly independent on an interval $[a, b]$ if

(i) $f_1(t)f_2(t) - g_1(t)g_2(t) \neq 0,\ \forall\ t \in [a, b]$

(ii) $f_1(t)g_1(t) - f_2(t)g_2(t) \neq 0, \ \forall \ t \in [a, b]$

(iii) $f_1(t)g_2(t) - f_2(t)g_1(t) \neq 0, \ \forall \ t \in [a, b]$

(iv) $f_1(t)g_2(t) + f_2(t)g_1(t) = 0, \ \forall \ t \in [a, b]$

(i) Which of the following is correct?

(i) A hyperbolic equation has one real family of characteristics.

(ii) For an elliptic equation there does not exist any real characteristics.

(iii) For a parabolic equation there does not exist any real characteristics.

(iv) An elliptic equation has two real and distinct families of charactersitics.

(j) The order of the partial differential equation

$$\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial y^3}\right\}^{\frac{1}{2}} = 2x\left(\frac{\partial^2 u}{\partial x^2}\right)^2$$

is

(i) 3

(ii) 1

(iii) 4

(iv) 2

(Continued)

2. Answer the following questions (any *five*) : 2×5=10

(a) Find the partial differential equation arising from the surface

$$z = xy + f(x^2 + y^2)$$

(b) Give geometrical interpretation of first-order linear partial differential equation.

(c) If D is a differential operator with respect to t, then find $(D^2 + 1)(3D + 2)t^3$.

(d) Show that the family of spheres $x^2 + y^2 + (z - c)^2 = \pi^2$ satisfies the first-order linear partial differential equation $yp - xq = 0$.

(e) Determine the regions in which the equation $u_{xx} + y^2 u_{yy} = y$ is parabolic and elliptic.

(f) Write the normal form of a linear system of n differential equations in n unknown functions $x_1, x_2, x_3, \ldots, x_n$.

(g) Transform the single linear differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = 0$$

into a system of first-order differential equations.

(Turn Over)

3. Answer the following questions (any *five*) : 5×5=25

(a) Find the general solution of the linear equation $x^2u_x + y^2u_y = (x+y)u$.

(b) Applying the method of separation of variables, solve the following equation :

$$u_x + 2u_y = 0, \quad u(0, y) = 3e^{-2y}$$

(c) Solve the following system of equations by using operator method :

$$\frac{dx}{dt} + \frac{dy}{dt} - x - 3y = e^t$$
$$\frac{dx}{dt} + \frac{dy}{dt} + x = e^{3t}$$

(d) Show that $u_1 = e^x$ and $u_2 = e^{-y}$ are solutions of the non-linear equation $(u_x + u_y)^2 - u^2 = 0$ but their sum $(e^x + e^{-y})$ is not a solution of this equation.

(e) Derive one-dimensional heat equation in the form

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(f) Determine the integral surface of the equation

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the data $x + y = 0$, $u = 1$.

(g) Find the general solution of the following system of equations :

$$\frac{dx}{dt} = 3x + y$$
$$\frac{dy}{dt} = 4x + 3y$$

(h) Find the first three approximations of the solution by using the method of successive approximations in the following initial-value problem

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0$$

(i) Find the solution of the initial-value systems

$$u_t + 3uu_x = v - x, \qquad v_t - cv_x = 0$$

with $u(x, 0) = x$ and $v(x, 0) = x$.

4. Answer the following questions (any *two*) : 10×2=20

(a) Find the characteristic equations and characteristic curves, and then reduce the equation

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$$

to canonical form. 2+1+7=10

(b) Find the solution of

$$u_{xx} - u_{yy} = 1$$
$$u(x, 0) = \sin x$$
$$u_y(x, 0) = x$$

(c) Applying the modified Euler method to

$$\frac{dy}{dx} = x^2 + y^2$$
$$y(0) = 1$$

approximate the values of the solution y at $x = 0 \cdot 1$ and $0 \cdot 2$, using $h = 0 \cdot 1$. Obtain the results to three figures after the decimal point.

(d) Solve the initial boundary-value problem

$$u_t = ku_{xx}, \quad 0 < x < l, \quad t > 0$$
$$u(0, t) = 0, \quad t \geq 0$$
$$u(l, t) = 0, \quad t \geq 0$$
$$u(x, 0) = x(l - x), \quad 0 \leq x \leq l$$

by using the method of separation of variables.

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