63/1 (SEM-3) CC5/MATHC3056

2023

MATHEMATICS

Paper : MATHC3056

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer from the following (any *six*) : 1×6=6

(a) For the open interval $A = (0, 1)$, which of the following is correct?

(i) 0 is not the cluster point of A

(ii) Only 0 and 1 are cluster points of A

(iii) Every point of the closed interval $[0, 1]$ is a cluster point of A

(iv) A has no cluster points

(b) $\lim_{x\to 0} \frac{a^x - 1}{x}$ where $a > 0$ is equal to

(i) 0

(ii) 1

(iii) $\log a$

(iv) e^a

(c) Let I be an interval and $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is

(i) a finite set

(ii) an empty set

(iii) an interval

(iv) None of the above

(d) If $f: I \to \mathbb{R}$ be a continuous function at the point c, then

(i) c is a point of the domain

(ii) $f(c)$ must exist

(iii) $\lim_{x\to c} f(x) = f(c)$

(iv) $\lim_{x\to c} f(x) \neq f(c)$

(e) Lagrange's mean value theorem can be proved for a function $f(x)$ by applying Rolle's theorem to the function

(i) $\phi(x) = f(x) + kx^2$

(ii) $\phi(x) = f(x) - kx^2$

(iii) $\phi(x) = f(x) + kx$

(iv) $\phi(x) = \{f(x)\}^2 + kx^2$

(f) The necessary condition for a function $f(x)$ to have a maxima at $x = c$ is

(i) $f'(c) > 0$

(ii) $f'(c) = 0$

(iii) $f'(c) < 0$

(iv) None of the above

(g) The $(n+1)$th term in Maclaurin series for the function $f(x)$ is

(i) $\frac{x^n}{n} f^n(x)$

(ii) $\frac{x^n}{n!} f^n(x)$

(iii) $\frac{x^n}{n!} f^n(0)$

(iv) $f^n(0)$

(h) If $(x) = |x|$, then

(i) $f'(0) = 0$

(ii) $f(x)$ is maximum at $x = 0$

(iii) $f(x)$ is minimum at $x = 0$

(iv) None of the above

(i) The function $f(x) = \begin{cases} 1+x & \text{if } x \le 2 \\ 5-x & \text{if } x > 2 \end{cases}$ is

(i) continuous for all values of x

(ii) continuous for all values of x except at $x = 2$

(iii) discontinuous at $x = 1$

(iv) discontinuous at $x = 5$

(j) A function $f(x)$ is said to be differentiable at $x = c$

(i) if $Rf'(c)$ and $Lf'(c)$ both exist

(ii) if only $Rf'(c)$ exist

(iii) if $Lf'(c)$ do not exist

(iv) if $Rf'(c)$ and $Lf'(c)$ both exist and are equal

2. Answer any *five* of the following questions : $2 \times 5 = 10$

(a) Define ε-δ definition on limit at a point c.

(b) Let f and g be defined on A to $\mathbb{R}$ and c be a cluster point of A. If $\lim_{x \to c} f$ and $\lim_{x \to c} fg$ exist, does it follow that $\lim_{x \to c} g$ exists? Justify your answer by an example.

(c) Evaluate $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} (x > 0)$.

(d) Show that the function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(e) Check whether the equation $f(x) = xe^x - 2$ has a root in the interval $[0, 1]$.

(f) Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(g) Find the points of relative extrema for the functions $f(x) = x^2 - 3x + 5$, defined on $\mathbb{R}$ to $\mathbb{R}$.

3. Answer any *six* parts from the following questions : 5×6=30

(a) Use ε-δ definition of a limit to prove $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$.

(b) Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Prove that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c} |f(x) - L| = 0$.

(c) Prove that

(i) $\lim_{x \to 0^+} \frac{1}{x} = \infty$

(ii) $\lim_{x \to \infty} \frac{1}{x^2} = 0$

(d) Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighbourhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.

(e) Prove that the following function :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any point of $\mathbb{R}$.

(f) State squeeze theorem. Use it to show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. 2+3=5

(g) Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that—

(i) the sum $f+g$ is continuous at c

(ii) the product fg is continuous at c

(h) Show that $f(x)=\sin x$ is continuous on $\mathbb{R}$.

(i) Determine where the function $f(x)=|x|+|x-1|$ from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative.

(j) State and prove Caratheodory theorem.

4. Answer any *two* parts from the following : 10×2=20

(a) *(i)* Functions f and g are defined on $\mathbb{R}$ by

$$f(x)=x+1 \text{ and } g(x)=\begin{cases}2, & \text{if } x\neq 1\\ 0, & \text{if } x=1\end{cases}$$

Find $\lim_{x\to 1} g(f(x))$ and compare with the value of $g(\lim_{x\to 1} f(x))$. 3+3=6

(ii) Let $n\in\mathbb{N}$ be such that $n\geq 3$. Derive the inequality $-x^2\leq x^n\leq x^2$ for $-1<x<1$. Then use the fact that $\lim_{x\to 0} x^2=0$ to show that $\lim_{x\to 0} x^n=0$. 4

(b) *(i)* Show that the function $f(x)=\frac{1}{x^2}$ is uniformly continuous on the set $A=[1,\infty)$, but that is not uniformly continuous on $B=(0,\infty)$. 6

(ii) Show that if f and g are uniformly continuous on $A\subseteq\mathbb{R}$, then $f+g$ is uniformly continuous on A. 4

(c) *(i)* State prove Rolle's theorem. 5

(ii) Suppose that f is continuous on the closed interval $[a, b]$, that f is differentiable on the open interval (a, b) and that $f'(x)=0$ for $x\in(a, b)$. Then show that f is constant on $[a, b]$. 5

(d) State and prove Location of Roots theorem and use this theorem to show that the equation $x = \cos x$ has a solution in

$$\left[0, \frac{\pi}{2}\right]$$

2+6+2=10

5. Answer any *one* part from the following : 14

(a) State and prove Darboux's theorem : Suppose that if $f:[0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, and that $f(0)=0$, $f(1)=1$, $f(2)=1$

(i) Show that there exists $c_1 \in (0, 1)$ such that $f'(c_1)=1$.

(ii) Show that there exists $c_2 \in (1, 2)$ such that $f'(c_2)=0$.

(iii) Show that there exists $c \in (0, 2)$ such that $f'(c)=\frac{1}{3}$. 2+4+2+3+3=14

(b) (i) State Maclaurin's series expansion for the function $f(x)$ and obtain the expansion of the function $f(x)=e^x$, showing the convergence of the remainder term after n terms. 2+5=7

(ii) Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathbb{R}$, $L \neq 0$, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

Show that if $L > 0$, then $\lim_{x\to\infty} f(x) = \infty$ if and only if $\lim_{x\to\infty} g(x) = \infty$. 7

(c) (i) Let $h: \mathbb{R} \to \mathbb{R}$, defined by $h(x) = x^3 + 2x + 1$. Show that h is continuous and strictly monotonic increasing on $\mathbb{R}$. Further deduce that h^{-1} is differentiable on $\mathbb{R}$. Also find the value $(h^{-1})'(h(1))$.

1+2+3+3=9

(ii) Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is negative on I, then f is strictly decreasing on I. 5

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