Chapter 2

Five Dimensional Kaluza-Klein Dark Energy Cosmological Model under Two-Fluid Scenario with Variable Deceleration Parameter

2.1 Introduction

Many cosmological observations, such as the Type Ia Supernovae experiment (SNe.Ia) have suggested that DE is the reason for the accelerating expansion of the observable universe. There are many theoretical models of dark energy's available today. The best explanation model for DE is the ΛCDM model. According to cosmological observations and analysis, DE accounts for roughly 68.3%, dark matter accounts for approximately 26.7%, and baryonic matter accounts for approximately 5% of the total energy of the universe, with negligible radiation.

The evolution of the DE parameter in the two-fluid scenario has been studied by several eminent authors. The cosmology of viscous dark tachyons in a non-flat FRW universe has been investigated by Setare et al. (2009). Thermodynamics of DE

The work presented in this chapter has been published in the journal *"Journal of Mathematical and Computational Science"* (2021), **11** (6): 7699-7716, https://doi.org/10.28919/jmcs/6382.

interaction between dark matter and radiation are studied by Jamil et al. (2010). The authors Amirhashchi et al. (2011a,b) have studied the development of the DE parameter in an isotropic and spatially homogenous FRW space-time filled with barotropic fluid and DE by taking a time dependent deceleration parameter. The DE models in the FRW universe with a constant deceleration parameter has been investigated by Pradhan et al. (2011). The interacting and non-interacting two-fluid scenario for DE models in anisotropic Bianchi type-I space-time has been studied by Singh and Chaubey (2012). Saha et al. (2012) have investigated two-fluid scenario for DE models in the FRW universe. DE in the scalar tensor theory of gravity is described as a two-fluid scenario by Reddy and Kumar (2013). Two-fluid cosmological models in a 5D spherically symmetric space-time are investigated by Samanta and Debata (2013). Pradhan (2014) have investigated FRW DE models under two-fluid scenario from decelerating to accelerating phase of expansion of the universe. In a spatially homogenous and isotropic FRW universe with a time-dependent q, Amirhashchi et al. (2014) explored two-fluid atmosphere for DE models. Rao et al. (2016) have studied Axially Symmetric DE cosmological models under two fluid scenario in Brans-Dicke theory of gravitation. Interacting and non-interacting Bianchi Type-II, VIII, and IX DE cosmological models under two-fluid scenario of Brans-Dicke theory are studied by Rao and Sireesha (2017). Tripathy et al. (2017) investigated two fluid anisotropic Bianchi type-V DE models in a scale invariant theory. Tiwari et al. (2017) investigated two-fluid scenario DE cosmological model in FRW universe. In Bianchi type-III space time filled with a barotropic fluid and DE, Tiwari et al. (2018) investigated into the EoS parameter for DE by considering a variable deceleration parameter. The Bianchi type-I cosmological model with barotropic and DE type fluids was studied by Goswami et al. (2020). The stability of DE cosmological models in an anisotropic Bianchi type-V space time was investigated by Mishra et al. (2021). Kumar et al. (2022) have investigated on two-fluid cosmological models that include matter and a radiating source in the context of the Saez-Ballester scalar-tensor theory of gravitation. The Bianchi type-I cosmological model under two fluids scenario in scale covariant theory of gravitation has been studied by Hatkar et al. (2022). In Saez-Ballester theory of gravitation, Trivedi and Bhabor (2022) studied the characteristics of the five-dimensional Bianchi type-I cosmic universe filled with barotropic fluid and DE.

Inspired by the above mentioned recent efforts of many authors, we have explored the evolution of the DE parameter in a 5D Kaluza-Klein space-time with barotropic fluid and DE. We have discussed the behaviors of the models in presence of two fluid scenario. We also obtained the value of the statefinder parameter in the proposed model.

2.2 Metric and Field Equations

A spatially homogeneous and anisotropic 5D Kaluza-Klein space time has been considered and it is as follows:

$$ds^2 = dt^2 - A^2(dx^2 + dy^2 + dz^2) - B^2 d\psi^2 \tag{2.1}$$

where A and B are functions of cosmic time t only and the fifth coordinate ψ is taken to be extended space like coordinate.

Einstein's field equation (with $8\pi G = 1$ and $c = 1$) is given by

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij} \tag{2.2}$$

where R_{ij} is the Ricci tensor, R is the Ricci scalar, g_{ij} is the metric tensor and T_{ij} is the two fluid energy-momentum tensor consisting of barotropic fluid and dark energy.

The energy momentum tensor for two fluid is given by

$$T_{ij} = T_{ij}^m + T_{ij}^D \tag{2.3}$$

where

$$T_{ij}^m = (\rho_m + p_m)]\mu_i \mu_j - p_m g_{ij} \tag{2.4}$$

$$T_{ij}^D = (\rho_D + p_D)\mu_i \mu_j - p_D g_{ij} \tag{2.5}$$

here ρ_m and p_m are the energy density and pressure of the perfect fluid and ρ_D and p_D are the energy density and pressure of the dark energy component.

The EoS parameter (ω) which is considered as an important quantity in describing the dynamics of the universe, is the ratio of the pressure (p) and the energy density (ρ) and is given by

$$p_m = \omega_m \rho_m \tag{2.6}$$

and

$$p_D = \omega_D \rho_D \tag{2.7}$$

The Einstein field eqn. (2.2) with energy momentums eqns. (2.4) and (2.5) for the metric (2.1) it follows that

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} = -(p_m + p_D) \tag{2.8}$$

$$3\frac{\ddot{A}}{A} + 3\frac{\dot{A}^2}{A^2} = -(p_m + p_D) \tag{2.9}$$

$$3\frac{\dot{A}^2}{A^2} + 3\frac{\dot{A}\dot{B}}{AB} = \rho_m + \rho_D \tag{2.10}$$

an over dot indicate a derivatives with respect to cosmic time t.

The energy conservation equation $T^{ij}_{;j} = 0$, which yields

$$\dot{\rho} + 4H(\rho + p) = 0 \tag{2.11}$$

where $p = p_m + p_D$ and $\rho = \rho_m + \rho_D$.

2.3 Solution of the Field Equations

In order to solve the system of equations we consider the deceleration parameter (q) as a linear function of Hubble's parameter which is as follows

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \beta H + \alpha \tag{2.12}$$

Here α and β arbitrary constants.

Taking $\alpha = -1$ in eqn. (2.12)

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \beta H \tag{2.13}$$

which yields the following differential equation

$$\frac{a\ddot{a}}{\dot{a}^2} + \beta\frac{\dot{a}}{a} - 1 = 0 \tag{2.14}$$

which on integration gives

$$a(t) = e^{\frac{1}{\beta}\sqrt{2\beta t + c}} \tag{2.15}$$

where c is an integrating constant.

Subtracting eqn. (2.8) from eqn. (2.9) we get,

$$\frac{d}{dt}(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}) + (\frac{\dot{A}}{A} - \frac{\dot{B}}{B})\frac{\dot{V}}{V} = 0$$

which on integrating gives

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{\lambda}{V} \tag{2.16}$$

where λ is an integration constant.

Using eqn. (2.15) in eqn. (2.16) and then integrating we get the scale factors are

$$A(t) = e^{\frac{1}{\beta}\sqrt{2\beta t + c}} exp[-\frac{\lambda\beta}{64} e^{\frac{-4}{\beta}\sqrt{2\beta t + c}}(\frac{4}{\beta}\sqrt{2\beta t + c} + 1)] \tag{2.17}$$

and

$$B(t) = e^{\frac{1}{\beta}\sqrt{2\beta t + c}} exp[\frac{3\lambda\beta}{64} e^{\frac{-4}{\beta}\sqrt{2\beta t + c}}(\frac{4}{\beta}\sqrt{2\beta t + c} + 1)] \tag{2.18}$$

Therefore the metric (2.1) reduces to

$$ds^2 = dt^2 - e^{\frac{1}{\beta}\sqrt{2\beta t + c}} exp[-\frac{\lambda\beta}{64} e^{\frac{-4}{\beta}\sqrt{2\beta t + c}}(\frac{4}{\beta}\sqrt{2\beta t + c} + 1)](dx^2 + dy^2 + dz^2)$$

$$- e^{\frac{1}{\beta}\sqrt{2\beta t + c}} exp[\frac{3\lambda\beta}{64} e^{\frac{-4}{\beta}\sqrt{2\beta t + c}}(\frac{4}{\beta}\sqrt{2\beta t + c} + 1)]d\phi^2 \tag{2.19}$$

The physical parameters of the model are given by

$$V = e^{\frac{4}{\beta}\sqrt{2\beta t + c}} \tag{2.20}$$

$$H = \frac{1}{\sqrt{2\beta t + c}} \tag{2.21}$$

$$\theta = 4H = \frac{4}{\sqrt{2\beta t + c}} \tag{2.22}$$

$$\Delta = \frac{3\lambda^2}{16}(2\beta t + c)e^{\frac{-8}{\beta}\sqrt{2\beta t + c}} \tag{2.23}$$

$$\sigma^2 = \frac{3\lambda^2}{32}e^{\frac{-8}{\beta}\sqrt{2\beta t + c}} \tag{2.24}$$

$$q = -1 + \frac{\beta}{\sqrt{2\beta t + c}} \tag{2.25}$$

Here, we consider two cases:

Case-I: when the two fluids do not-interact with each other and

Case-II: when the two fluids interact each other.

2.4 Case-I: Non-Interacting Two Fluid Model

First we consider the two fluids do not interact, the conservation equation for the dark and barotropic fluid separately as

$$\dot{\rho}_m + 4H(p_m + \rho_m) = 0 \tag{2.26}$$

$$\dot{\rho}_D + 4H(p_D + \rho_D) = 0 \tag{2.27}$$

Integrating (2.26) we get,

$$\rho_m = \rho_0 e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t + c}} \tag{2.28}$$

where ρ_0 is an integrating constant.

Using (2.17), (2.18) and (2.28) in eqns. (2.9) and (2.10) we obtain

$$P_D = -3\left[\frac{2+\frac{\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \beta(2\beta t+c)^{\frac{-1}{2}}}{(2\beta t+c)}\right] - \omega_m\rho_0 e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}} \tag{2.29}$$

$$\rho_D = 3\left[\frac{2-\frac{\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}}}{(2\beta t+c)}\right] - \rho_0 e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}} \tag{2.30}$$

Using eqns. (2.29) and (2.30) we can find the expression for EoS parameter of dark energy as

$$\omega_D = -\left[\frac{6+\frac{3\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - 3\beta(2\beta t+c)^{\frac{-1}{2}} + \omega_m\rho_0(2\beta t+c)e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}}}{6-\frac{3\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \rho_0(2\beta t+c)e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}}}\right] \tag{2.31}$$

The expressions of matter-energy-density parameter (Ω_m) and dark energy density parameter (Ω_D) are given by

$$\Omega_m = \frac{\rho_m}{6H^2} = \frac{\rho_0 e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}}}{6(2\beta t+c)^{-1}} \tag{2.32}$$

and

$$\Omega_D = \frac{\rho_D}{6H^2} = 1 - \frac{\lambda^2}{16}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \frac{1}{6}\rho_0(2\beta t+c)e^{\frac{-4(1+\omega_m)}{\beta}\sqrt{2\beta t+c}} \tag{2.33}$$

The total energy density (Ω) is given by

$$\Omega = \Omega_m + \Omega_D = 1 - \frac{\lambda^2}{16}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} \tag{2.34}$$

2.5 Case-II: Interacting Two Fluid Model

In this section, we can write the energy conservation equation for the dark and barotropic fluid as

$$\dot{\rho}_m + 4(1+\omega_m)\rho_m H = Q \tag{2.35}$$

$$\dot{\rho}_D + 4(1+\omega_D)\rho_D H = -Q \tag{2.36}$$

where the quantity Q represents the interaction between the componants of matter and DE. We considerd $Q > 0$, this ensures that the energy is being transferred from DE to the matter component. Following Guo et al. (2007), Amendola et al. (2007), we consider

$$Q = 4Hk\rho_m \tag{2.37}$$

where k is coupling constant.
Using (2.37) in eqn. (2.35) we get

$$\rho_m = \rho_0 e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c}} \tag{2.38}$$

Now the value of P_D and ρ_D are

$$P_D = -3\left[\frac{2+\frac{\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \beta(2\beta t+c)^{\frac{-1}{2}}}{(2\beta t+c)}\right] - \omega_m\rho_0 e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c}} \tag{2.39}$$

$$\rho_D = 3\left[\frac{2-\frac{\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}}}{(2\beta t+c)}\right] - \rho_0 e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c}} \tag{2.40}$$

Also the EoS parameter (ω_D) is obtained as

$$\omega_D = -\left[\frac{6+\frac{3\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - 3\beta(2\beta t+c)^{\frac{-1}{2}} + \omega_m\rho_0(2\beta t+c)e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c}}}{6-\frac{3\lambda^2}{8}(2\beta t+c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \rho_0(2\beta t+c)e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c}}}\right] \tag{2.41}$$

The expression of matter energy density-parameter (Ω_m) and dark energy density parameter (Ω_D) are given as

$$\Omega_m = \frac{\rho_m}{6H^2} = \frac{\rho_0 e^{\frac{-4(1+\omega_m-k)}{\beta}\sqrt{2\beta t+c_1}}}{6(2\beta t+c_1)^{-1}} \tag{2.42}$$

and

$$\Omega_D = \frac{\rho_D}{6H^2} = 1 - \frac{\lambda^2}{16}(2\beta t + c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} - \frac{1}{6}\rho_0(2\beta t + c)e^{\frac{-4(1+\omega_m - k)}{\beta}\sqrt{2\beta t+c}} \quad (2.43)$$

The total energy density (Ω) is given by

$$\Omega = \Omega_m + \Omega_D = 1 - \frac{\lambda^2}{16}(2\beta t + c)e^{\frac{-8}{\beta}\sqrt{2\beta t+c}} \quad (2.44)$$

For the proposed model, statefinder parameter (r, s) are obtained as

$$r = 1 - \frac{3\beta}{\sqrt{2\beta t + c}} + \frac{3\beta^2}{2\beta t + c} \quad (2.45)$$

and

$$s = \frac{-2\beta\sqrt{2\beta t + c} + 2\beta^2}{2\beta\sqrt{2\beta t + c} - 3(2\beta t + c)} \quad (2.46)$$

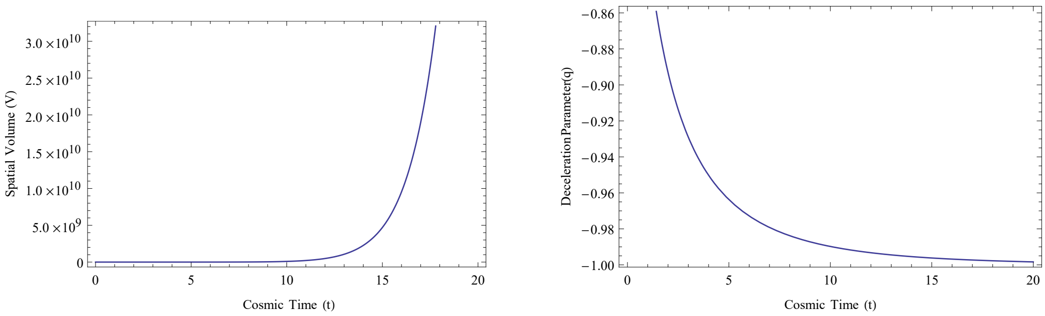
From the above result it is observed that as $t \to \infty$, $(r, s) \to (1, 0)$ which gives that the model tends to the ΛCDM model as in recent observation (Ahmed and Pradhan (2014)).

2.6 Physical Interpretation of the Solutions

Here, we discuss the outcomes of the above mentioned theoretical calculations. We have plotted the various figures using these calculations for suitable values of the constants mentioned in the corresponding captions.

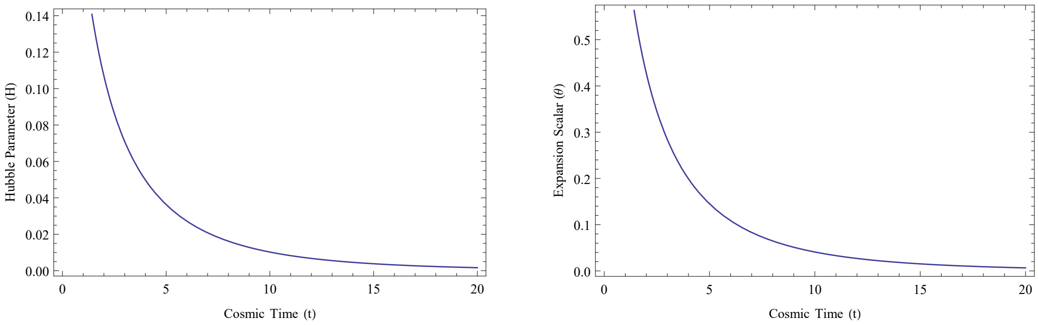
From the expression (2.20), we observed that the behavior of spatial volume V is zero at initial epoch $t = 0$ and it is increasing expontially with respect to cosmic time t which shows that the model universe is expanding with the evolution of time. At time $t \to \infty$, the spatial volume V becomes infinite. Nature of the variations of spatial volume V versus cosmic time t is plotted in fig. 2.1 (a).

From eqn. (2.25) we observed that the universe is in an accelerating phase when $q < 0$ for $t > \frac{\beta^2 - c}{2\beta}$. Also, current observations of Type Ia Supernovae (Riess et al. (1998); Garnavich et al. (1998); Schmidt et al. (1998); Perlmutter et al. (1999)) exposed that the present universe is accelerating and the value of deceleration parameter q can be



(a) (b)

Figure 2.1: (a) The plot of V versus cosmic time t, (b) The plot of deceleration parameter q versus cosmic time t, for $\beta = 1$ and $c = 1$.



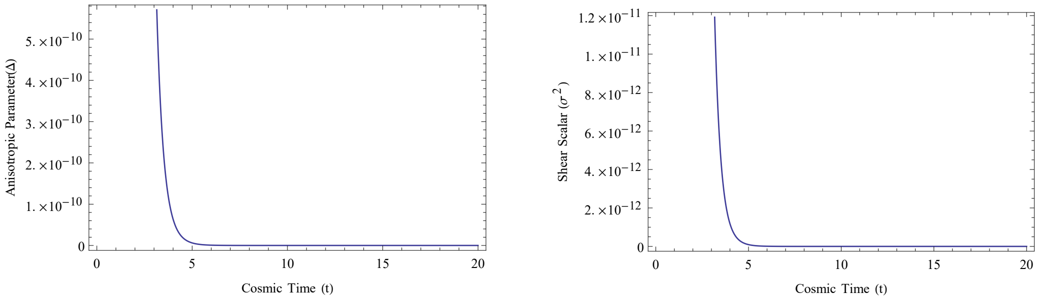
(a) (b)

Figure 2.2: (a) The plot of H versus cosmic time t, (b) The plot of expansion scalar θ versus cosmic time t, for $\beta = 1$ and $c = 1$

found nearby in the range $-1 < q < 0$. Variations of q versus cosmic time t is plotted in fig. 2.1 (b). From this figure it is observed that $q < 0$, i.e., at present time we obtain the accelerating phase of the universe. As $t \to \infty$, the models asymptotically approach the value $q = -1$, corresponding to the ΛCDM model.

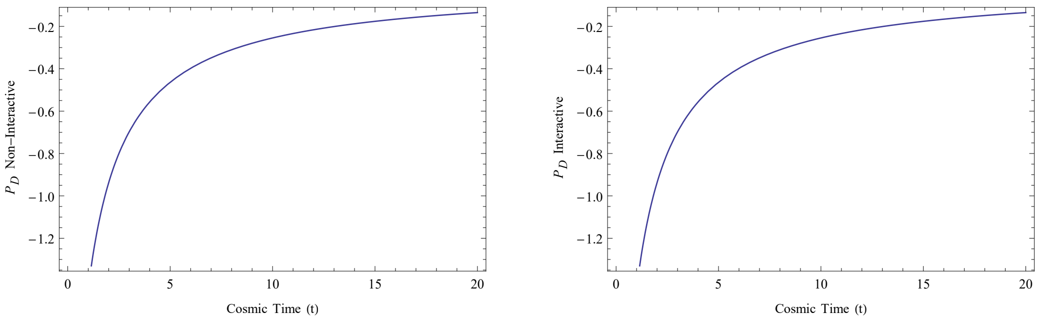
From eqns. (2.21) and (2.22) shows that both Hubble's parameter H and expansion scalar θ are always positive and decreasing function of cosmic time t and tends to zero as $t \to \infty$. Variations of Hubble's parameter H and expansion scalar θ versus cosmic time t are depicted in fig. 2.2 (a) and fig. 2.2 (b), which are in agrees with the prevailing theories.

From figs. 2.3 (a) and 2.3 (b) corresponding to the eqns. (2.23) and (2.34), it is



(a) (b)

Figure 2.3: (a) The plot of Δ versus cosmic time t, (b) The plot of shear scalar σ^2 versus cosmic time t, for $\beta = 1$, $\lambda = 1$ and $c = 1$.

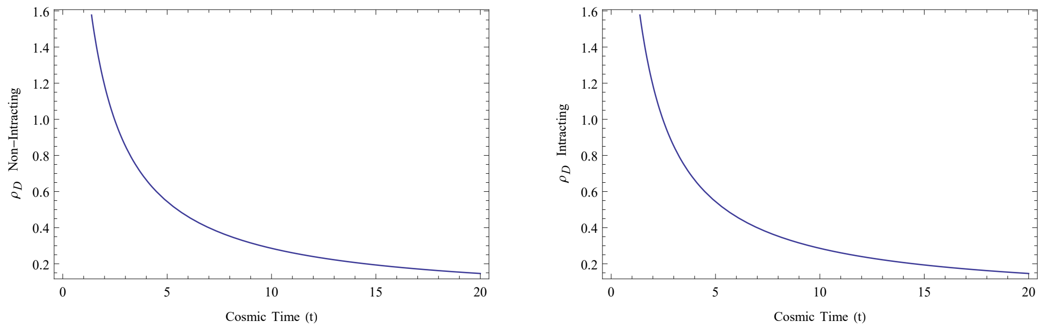


(a) (b)

Figure 2.4: (a) The plot of p_D versus cosmic time t for non- interactive case, (b) The plot of pressure p_D versus cosmic time t for interactive case, for $\beta = 1$,$c = 1$,$\lambda = 1$, $\rho_0 = 1$ and $\omega_m = 0.5$.

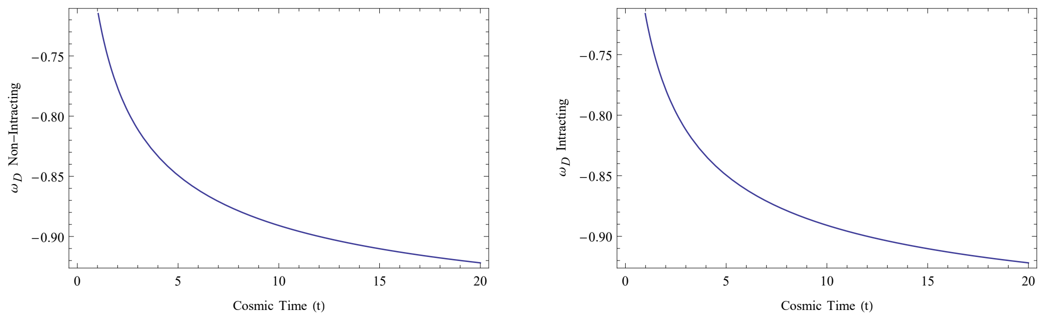
seen that the anisotropic parameter Δ and shear scalar σ^2 tends to zero as cosmic time $t \to \infty$.

Figs. 2.4 (a) and 2.4 (b) shows that the variation of pressure p_D versus cosmic time t for both the cases. We observed that at the time of evolution of the universe $(t = 0)$, the pressure p_D is negative and gradually increases, but remains negative. Nature of the variation of energy density ρ_D versus cosmic time t for both the cases are depicted in figs. 2.5 (a) and 2.5 (b). It is observed that the energy density ρ_D is a positive decreasing function of cosmic time t and becomes zero as $t \to \infty$. This negative pressure and positive energy density shows that the model under consideration represents a dark energy universe which shows accelerating expansion, as established



(a) (b)

Figure 2.5: (a) The plot of ρ_D versus cosmic time t for non-interactive case, (b) The plot of energy density ρ_D versus cosmic time t, for interactive case, for $\beta = 1$,$c = 1$,$\lambda = 1$, $\rho_0 = 1$, $k = .02$ and $\omega_m = 0.5$.



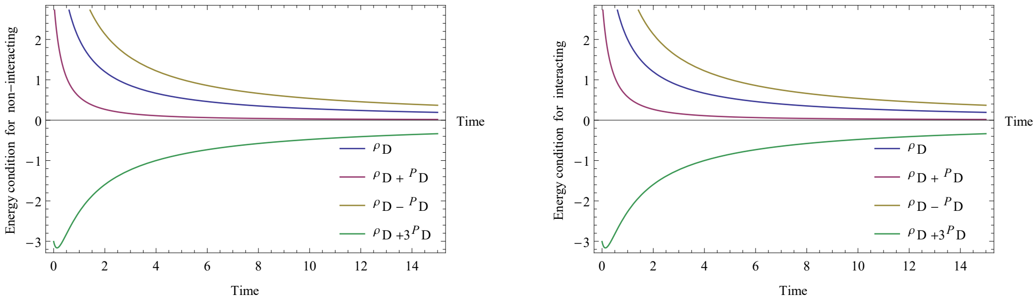
(a) (b)

Figure 2.6: (a) The plot of ω_D versus cosmic time t for non-interactive case, (b) The plot of ω_D versus cosmic time t for interactive case, for $\beta = 1$,$c = 1$,$\lambda = 1$, $\rho_0 = 1$, $k = .02$ and $\omega_m = 0.5$

by current observations (Perlmutter et al. (1999); Riess et al. (2004)).

From figs. 2.6 (a) and 2.6 (b) corresponding to the eqns. (2.31) and (2.41), we see that during the evolution of the universe for both non-interacting and interacting cases the behavior of EoS parameter ω_D is a decreasing function of cosmic time t. It is observed that, for both the cases it is same within the plotted range and varying in the quintessence model ($-1 < \omega_D < -\frac{1}{3}$) of the universe. Thus, the derived model is consistent with well established theoretical results as well as with the current observational data (Knop et al. (2003); Tegmark et al. (2004)).

From eqns. (2.34) and (2.44) we observed that in non-interacting case the total energy density parameter has the same properties as in interacting case. We observed that, for both interacting and non-interacting cases Ω approaches to one as $t \rightarrow \infty$.



(a) (b)

Figure 2.7: (a) The plot of energy condition versus cosmic time t for non-interactive case, (b) The plot of energy condition versus cosmic time t for interactive case, for $\beta = 1$,$c = 1$,$\lambda = 1$, $\rho_0 = 1$, $k = .02$ and $\omega_m = 0.5$.

Finally, in figs. 2.7 (a) and 2.7 (b) we have plottted the graphs of energy conditions such as weak energy condition (WEC), null energy condition (NEC), dominant energy condition (DEC) and strong energy condition (SEC) for both non-interacting and interacting cases . We observed that WEC ($\rho_D \geq 0$ and $\rho_D + P_D \geq 0$), NEC ($\rho_D + P_D \geq 0$) and DEC ($\rho_D - P_D \geq 0$) are satisfied for both the cases but the SEC ($\rho_D + 3P_D \leq 0$) gets violated, which indicates that the model proceed to accelerating expansion of the universe (Barcelo and Visser (2002); Riess et al. (2004)).