Chapter 5

Five Dimensional Kaluza-Klein Cosmological Model with Variable G and Λ in Conharmonically Flat Space

5.1 Introduction

The gravitational constant G and the cosmological constant Λ are two crucial parameters in Einstein's field equations. As we know, in the year 1917, Einstein first proposed the cosmological constant Λ as the force of universal repulsion in order to enable the static homogeneous solution of the Einstein's field equations in the existence of matter in accordance with the prevailing accepted theory at that time . The cosmological constant Λ is one of the most remarkable and unresolved problem in cosmology. After a few years, several researchers realised that the cosmological constant might be measured as the vacuum's energy density, which is the state of lowest energy, more than the vacuum's energy-momentum tensor $T_{ij}^{vac} = -\rho^{vac} g_{ij}$ and vacuum may also be considered as perfect fluid with equation of state (EoS) as $P_{vac} = -\rho^{vac} g_{ij}$ by taking $\rho_{vac} = \rho_\Lambda = \frac{\Lambda}{8\pi G}$ and moving with Λg_{ij} one can state that the effect of the energy momentum tensor in a vacuum is identical to Λ. Theoretical and experimental observations, such as the high red-shift Type Ia Supernovae experiment (SNe.Ia) (Perlmutter et al. (1998); Riess et al. (1998); Perlmutter et al. (1999)) has been established that the present universe is in an accelerated stage of expansion, and dark energy plays a vital role in driving this accelerated expansion (Copeland et al. (2006)). According to the latest PLANK 2013 results (Ade et al. (2014)), dark energy accounts for around 68.3% of the total content of the universe.

A rank four tensor L^l_{ijk} that retains its invariant form under conharmonic transformation for an n-dimensional Riemannian differentiable manifold is given by

$$L^l_{ijk} = R^l_{ijk} - \frac{1}{n-2}(g_{ij}R^l_k - g_{ik}R^l_j + \delta^l_k R_{ij} - \delta^l_j R_{ik}), \tag{5.1}$$

where R^l_{ijk} and R_{ij} are the Riemannian curvature tensor and the Ricci tensor respectively. Equation (5.1) is known as the conharmonic curvature tensor. A space time in which R^l_{ijk} vanish at each point is called a conharmonically flat space time.

The conharmonic curvature tensor shows the deviation of the space time from conharmonic flatness. Ahsan and Siddiqui (2009) discussed the significance of the conharmonic curvature tensor in four dimensional space time in general relativity. Siddiqui and Ahsan (2010) also studied the nature of the conharmonic curvature tensor in perfect fluid in the space time context of the general theory of relativity. Tiwari and Singh (2015) studied the role of conharmonically flat FRW space-time in the presence of cosmological constant Λ by taking $\Lambda = 3\beta H^2$, where β is a constant. Tiwari (2016) investigated the solution of conharmonic curvature tensor in general relativity. They solved Einstein's field equations by using the law of variation of a cosmological constant with a^m, where a is a scale factor and m is a constant. Tiwari and Shrivastava (2017) have investigated FRW cosmological model for conharmonically flat space time. Einstein's field equations with variable cosmological term are solved by using variable deceleration parameter. Goyal et al. (2019) have discussed a new class model of an accelerating FRW universe with time dependent variable G and Λ in conharmonically flat space. Recently Pradhan et al. (2020) studied a new class of holographic dark energy models in conharmonically flat space. Einstein's field equations are solved by considering the cosmological scale factor in the form of a hybrid expansion law.

In this chapter, we have investigated the 5D Kaluza-Klein cosmological models with time-dependent gravitational and cosmological constants in conharmonically flat space with the help of a variable deceleration parameter (DP). This chapter has been organised as follows: In Sec. 5.2, the metric and Einstein's field equations have been presented. In Sec. 5.3, the solution of Einstein's field equations is obtained. In Sect. 5.4, we discussed the geometrical and physical interpretation of the results.

5.2 Metric and Field Equations

A spatially homogeneous and isotropic 5D Kaluza-Klein space-time is given by

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-kr^2)d\psi^2\right], \tag{5.2}$$

where $a(t)$ represents the cosmic scale factor, which depicts how these distances (scales) change in contracting or expanding the universe and k is curvature parameter, which describes geometry of the spatial section of space time with an open ($k = -1$), flat ($k = 0$) and closed ($k = 1$) universe and the fifth coordinate ψ is taken to be an extended space like coordinate respectively.

The Einstein's field equations with time-dependent G and Λ can be written as

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G T_{ij} - \Lambda g_{ij}, \tag{5.3}$$

where R_{ij} is the Ricci tensor R is the Ricci scalar g_{ij} is the metric tensor and T_{ij} is the energy momentum tensor of a perfect fluid given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{5.4}$$

where ρ and p are energy density and isotropic pressure of the cosmic fluid and u_i is the five velocity vector with $u_i u^i = 1$.

The conharmonic curvature tensor for a relativistic 5Dl space-time is given by

$$L^l_{ijk} = R^l_{ijk} - \frac{1}{3}(g_{ij}R^l_k - g_{ik}R^l_j + \delta^l_k R_{ij} - \delta^l_j R_{ik}), \tag{5.5}$$

For conharmonically flat space-time $L^l_{ijk} = 0$, we obtain

$$3R^l_{ijk} = g_{ij}R^l_k - g_{ik}R^l_j + \delta^l_k R_{ij} - \delta^l_j R_{ik}. \tag{5.6}$$

Contracting above equation with $j = l$ and obtaining summation over j, we obtain

$$R_{ik} = -\frac{1}{5} R g_{ik}. \tag{5.7}$$

For conharmonically flat space-time, using the above equation the Einstein's field equation (5.3) reduce to

$$R_{ij} = \frac{16}{7} \pi G T_{ij} - \frac{2}{7} \Lambda g_{ij}. \tag{5.8}$$

In co-moving coordinates system, for the flat $(k = 0)$ metric (5.2) , the energy momentum tensor (5.4) and Einstein's field equation (5.8) it follows that

$$8\pi G\rho - \Lambda = -14(\dot{H} + H^2) \tag{5.9}$$

$$8\pi G p + \Lambda = \frac{7}{2}(\dot{H} + 4H^2) \tag{5.10}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and over head dot denotes the derivatives with respect to cosmic time t.

The equation of state parameter (ω) which is considered as an important quantity in describing the dynamics of the universe in the ratio of the pressure (p) and the energy density (ρ) are given by

$$p = \omega \rho \tag{5.11}$$

In the field equation (5.8), the cosmological constant Λ accounts for vacuum energy with its energy density ρ_ν and pressure p_ν satisfying the equation of state

$$p_\nu = -\rho_\nu = -\frac{\Lambda}{8\pi G} \tag{5.12}$$

The critical density and the density parameters for matter and cosmological constant are defined as

$$\rho_c = \frac{6H^2}{8\pi G} \tag{5.13}$$

$$\Omega_M = \frac{\rho}{\rho_c} = \frac{8\pi G}{6H^2} \tag{5.14}$$

$$\Omega_\Lambda = \frac{\rho_\nu}{\rho_c} = \frac{\Lambda}{6H^2} \tag{5.15}$$

5.3 Solution of the Field Equations

The set of field equations (5.9) and (5.10) are the system of two linearly independent equations with five unknown parameters p, ρ, G, Λ and H.

So first, we assume a power-law relation between the gravitational constant G and the scale factor a as proposed by Chawla et al. (2012) and recently used Goyal et al. (2019)

$$G \propto a^m \tag{5.16}$$

where m is a constant. For the sake of mathematical simplicity, eqn. (5.16) may be written as

$$G = G_0 a^m \tag{5.17}$$

where G_0 is a positive constant.

Secondly, we assume that the deceleration parameter (q) as a linear function of Hubble's parameter (Tiwari et al. (2015); Tiwari et al. (2018); Sharma et al. (2019); Dubey et al. (2021)).

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \beta H + \alpha \tag{5.18}$$

Here α and β are arbitrary constants.

From eqn. (5.18) we have

$$\frac{a\ddot{a}}{\dot{a}^2} + \beta\frac{\dot{a}}{a} + \alpha = 0 \tag{5.19}$$

By solving the eqn. (5.19) we obtain

$$a(t) = e^{[-\frac{(1+\alpha)}{\beta}t - \frac{1}{(1+\alpha)} + \frac{l}{\beta}]} \tag{5.20}$$

provided $\alpha \neq -1$ and l denotes constant of integration. For $\alpha = 0$, we obtain $q = -1$.

We take $\alpha = -1$ in eqn. (5.18) we have

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \beta H \tag{5.21}$$

which yields the following differential equation

$$\frac{a\ddot{a}}{\dot{a}^2} + \beta\frac{\dot{a}}{a} - 1 = 0 \tag{5.22}$$

which on integration gives

$$a(t) = e^{\frac{4}{\beta}\sqrt{2\beta t + c}} \tag{5.23}$$

where c denotes constant of integration.

From eqn. (5.23) the deceleration parameter is determined as

$$q = -1 + \frac{\beta}{\sqrt{2\beta t + c}} \tag{5.24}$$

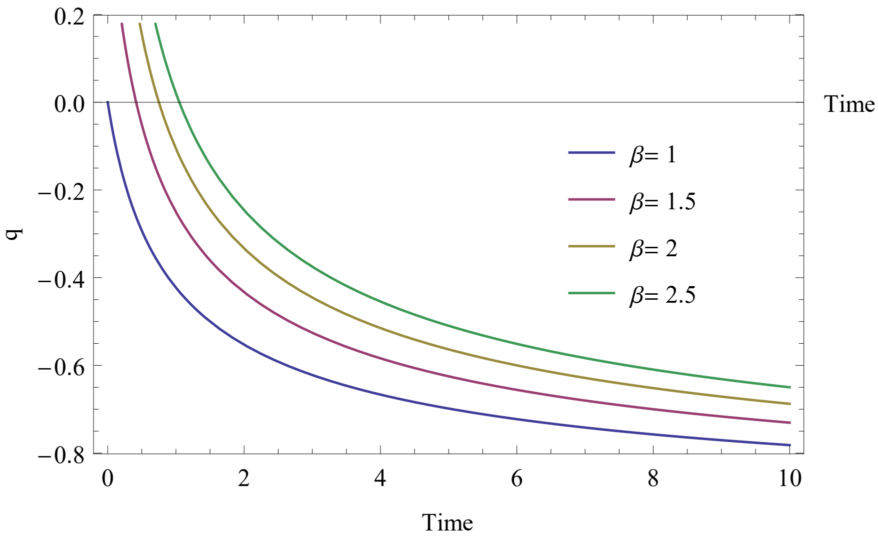

Figure 5.1: The plot of q versus cosmic time t, for $\beta = 1, 1.5, 2, 2.5$, $c = 1$.

There is a signature flipping in the deceleration parameter for the universe which was decelerating ($q > 0$ for $t < \frac{\beta^2 - c}{2\beta}$) in the past and has been accelerating ($q < 0$ for $t > \frac{\beta^2 - c}{2\beta}$) at present era (Sahoo et al. (2018);Goyal et al. (2019)). Also, current observations of Type Ia Supernovae (Riess et al. (1998); Garnavich et al. (1998); Schmidt et al. (1998); Perlmutter et al. (1999)) expose that the present universe is accelerating and the value of the deceleration parameter (q) can be found nearby in the range $-1 < q < 0$. Variation of q versus cosmic time t is plotted in Fig. 5.1.

Using (5.23) the model universe (5.2) becomes

$$ds^2 = dt^2 - e^{\frac{4}{\beta}\sqrt{2\beta t+c}}\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-kr^2)d\psi^2\right], \quad (5.25)$$

5.4 Physical Parameters of the Model and its Behaviour

For the cosmological model (5.25), the spatial volume (V), Hubble parameter (H), expansion scalar (θ), the gravitational constant (G), energy density (ρ), pressure(p), cosmological constant (Λ), vacuum energy density (ρ_ν), critical density (ρ_c) and the density parameter $(\Omega_M, \Omega_\Lambda)$ are obtained as follows:

$$V = e^{\frac{4}{\beta}\sqrt{2\beta t+c}} \quad (5.26)$$

$$H = \frac{1}{\sqrt{2\beta t + c}} \quad (5.27)$$

$$\theta = \frac{4}{\sqrt{2\beta t + c}} \quad (5.28)$$

$$G = G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}} \quad (5.29)$$

$$\rho = \frac{21\beta}{16\pi G_0(1+\omega)e^{\frac{m}{\beta}\sqrt{2\beta t+c}}(2\beta t + c)^{\frac{3}{2}}} \quad (5.30)$$

$$p = \frac{21\omega\beta}{16\pi G_0(1+\omega)e^{\frac{m}{\beta}\sqrt{2\beta t+c}}(2\beta t + c)^{\frac{3}{2}}} \quad (5.31)$$

$$\Lambda = -\frac{7\beta(1+4\omega)}{2(1+\omega)(2\beta t + c)^{\frac{3}{2}}} + \frac{14}{2\beta t + c} \quad (5.32)$$

$$p_\nu = -\rho_\nu = -\frac{1}{8\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}}[-\frac{7\beta(1+4\omega)}{2(1+\omega)(2\beta t+c)^{\frac{3}{2}}} + \frac{14}{2\beta t+c}] \tag{5.33}$$

$$\rho_c = \frac{3}{4\pi G_0(2\beta t+c)e^{\frac{m}{\beta}\sqrt{2\beta t+c}}} \tag{5.34}$$

$$\Omega_M = \frac{7\beta}{4(1+\omega)\sqrt{2\beta t+c}} \tag{5.35}$$

$$\Omega_\Lambda = -\frac{7\beta(1+4\omega)}{12(1+\omega)\sqrt{2\beta t+c}} + \frac{7}{3} \tag{5.36}$$

$$\Omega = \Omega_M + \Omega_\Lambda = \frac{7}{3} + \frac{7\beta(1-2\omega)}{6(1+\omega)\sqrt{2\beta t+c}} \tag{5.37}$$

The following are the three types of physical acceptable universes ($\omega = 0, 1, \frac{1}{3}$):

5.4.1 Matter Dominated Solution (Cosmology for $\omega = 0$)

For $\omega = 0$ in this case we obtained the expression for physical quantities as follows:

$$\rho = \frac{21\beta}{16\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}(2\beta t+c)^{\frac{3}{2}}} \tag{5.38}$$

$$p = 0 \tag{5.39}$$

$$\Lambda = -\frac{7\beta}{2(2\beta t+c)^{\frac{3}{2}}} + \frac{14}{2\beta t+c} \tag{5.40}$$

$$p_\nu = -\rho_\nu = -\frac{1}{8\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}}[-\frac{7\beta}{2(2\beta t+c)^{\frac{3}{2}}} + \frac{14}{2\beta t+c}] \tag{5.41}$$

$$\rho_c = \frac{3}{4\pi G_0(2\beta t+c)e^{\frac{m}{\beta}\sqrt{2\beta t+c}}} \tag{5.42}$$

$$\Omega_M = \frac{7\beta}{4\sqrt{2\beta t+c}} \tag{5.43}$$

$$\Omega_\Lambda = -\frac{7\beta}{12\sqrt{2\beta t + c}} + \frac{7}{3} \tag{5.44}$$

$$\Omega = \Omega_M + \Omega_\Lambda = \frac{7}{3} + \frac{7\beta}{6\sqrt{2\beta t + c}} \tag{5.45}$$

5.4.2 Zeldovich Fluid Distribution (Cosmology for $\omega = 1$)

For $\omega = 1$ in this case the expressions for physical quantities are obtained as follows:

$$\rho = \frac{21\beta}{32\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t + c}}(2\beta t + c)^{\frac{3}{2}}} \tag{5.46}$$

$$p = \frac{21\beta}{32\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t + c}}(2\beta t + c)^{\frac{3}{2}}} \tag{5.47}$$

$$\Lambda = -\frac{35\beta}{4(2\beta t + c)^{\frac{3}{2}}} + \frac{14}{2\beta t + c} \tag{5.48}$$

$$p_\nu = -\rho_\nu = -\frac{1}{8\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t + c}}}[-\frac{35\beta}{4(2\beta t + c)^{\frac{3}{2}}} + \frac{14}{2\beta t + c}] \tag{5.49}$$

$$\rho_c = \frac{3}{4\pi G_0 (2\beta t + c) e^{\frac{m}{\beta}\sqrt{2\beta t + c}}} \tag{5.50}$$

$$\Omega_M = \frac{7\beta}{8\sqrt{2\beta t + c}} \tag{5.51}$$

$$\Omega_\Lambda = -\frac{35\beta}{24\sqrt{2\beta t + c}} + \frac{7}{3} \tag{5.52}$$

$$\Omega = \Omega_M + \Omega_\Lambda = \frac{7}{3} - \frac{7\beta}{12\sqrt{2\beta t + c}} \tag{5.53}$$

5.4.3 Radiating Dominated Solution (Cosmology for $\omega = \frac{1}{3}$)

When $\omega = \frac{1}{3}$ we obtained the radiating dominated solution of the Einstein's field equation. In this case we obtained the expression for physical quantities as follows:

$$\rho = \frac{63\beta}{64\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}(2\beta t+c)^{\frac{3}{2}}} \tag{5.54}$$

$$p = \frac{21\beta}{64\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}(2\beta t+c)^{\frac{3}{2}}} \tag{5.55}$$

$$\Lambda = -\frac{21\beta}{8(2\beta t+c)^{\frac{3}{2}}} + \frac{14}{2\beta t+c} \tag{5.56}$$

$$p_\nu = -\rho_\nu = -\frac{1}{8\pi G_0 e^{\frac{m}{\beta}\sqrt{2\beta t+c}}}[-\frac{21\beta}{8(2\beta t+c)^{\frac{3}{2}}} + \frac{14}{2\beta t+c}] \tag{5.57}$$

$$\rho_c = \frac{3}{4\pi G_0(2\beta t+c)e^{\frac{m}{\beta}\sqrt{2\beta t+c}}} \tag{5.58}$$

$$\Omega_M = \frac{21\beta}{16\sqrt{2\beta t+c}} \tag{5.59}$$

$$\Omega_\Lambda = -\frac{49\beta}{64\sqrt{2\beta t+c}} + \frac{7}{3} \tag{5.60}$$

$$\Omega = \Omega_M + \Omega_\Lambda = \frac{7}{3} + \frac{35\beta}{64\sqrt{2\beta t+c}} \tag{5.61}$$

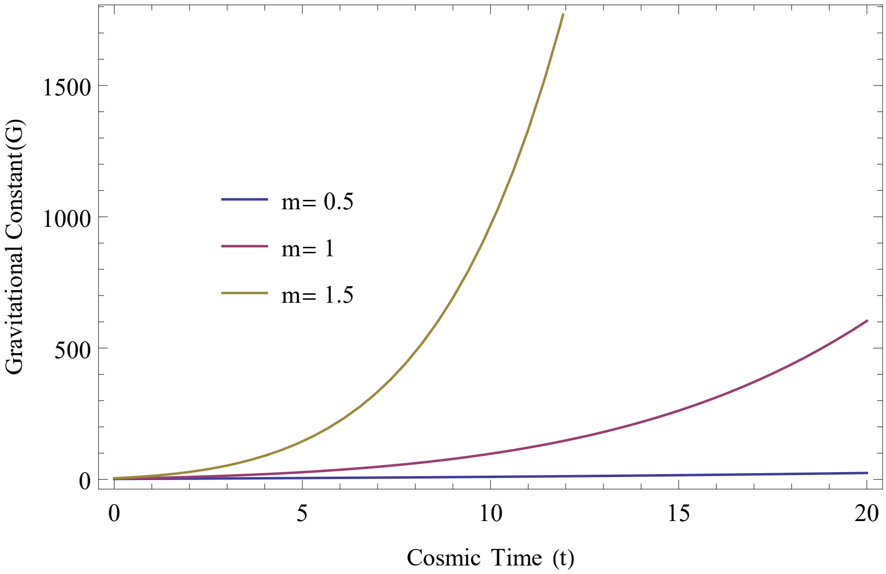

Figure 5.2: The plot of G versus cosmic time t, for $\beta = 1$,$m = 0.5, 1, 1.5$.

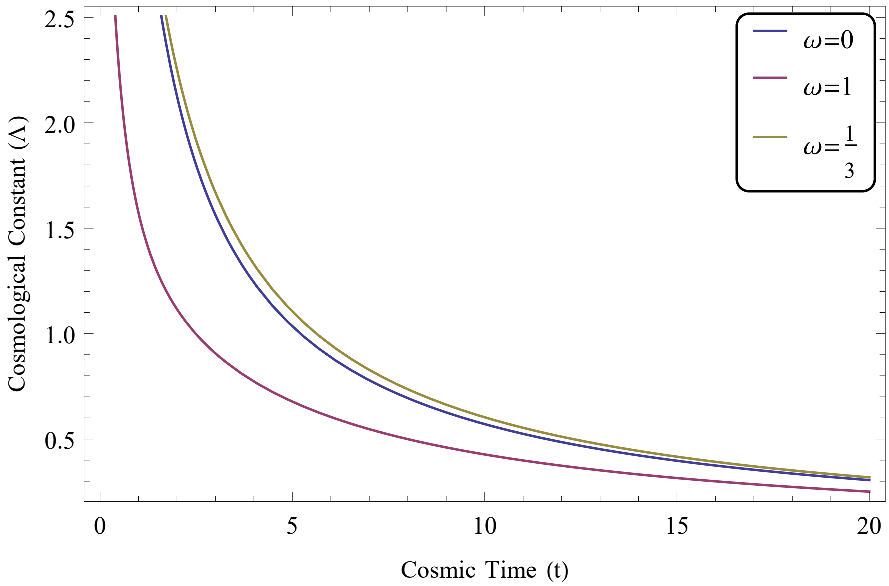

Figure 5.3: The plot of Λ versus cosmic time t, for $\beta = 1, \omega = 0, 1, \frac{1}{3}$.

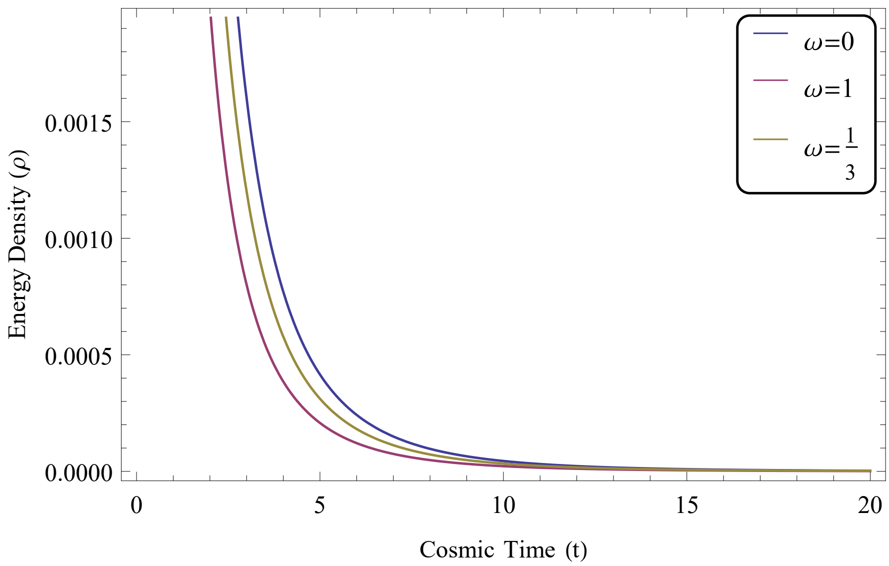

Figure 5.4: The plot of ρ versus cosmic time t, for $\beta = m = 1, \omega = 0, 1, \frac{1}{3}$.

From eqns. (5.27) and (5.28), it has been observed that the Hubble's parameter and expansion scalar both decreases as time increases . As $t \to \infty$ the Hubble's parameter and expansion scalar tends to a finite value. The derived model universe has a point type initial singularity (MacCallum (1971)). Also we observed that $\frac{dH}{dt}$ is negative which indicates that the model universe is expanding with an accelerated rate. The spatial volume is finite as $t = 0$ and it expands as time t increases and becomes infinity as $t \to \infty$, so the model represents an expanding universe.

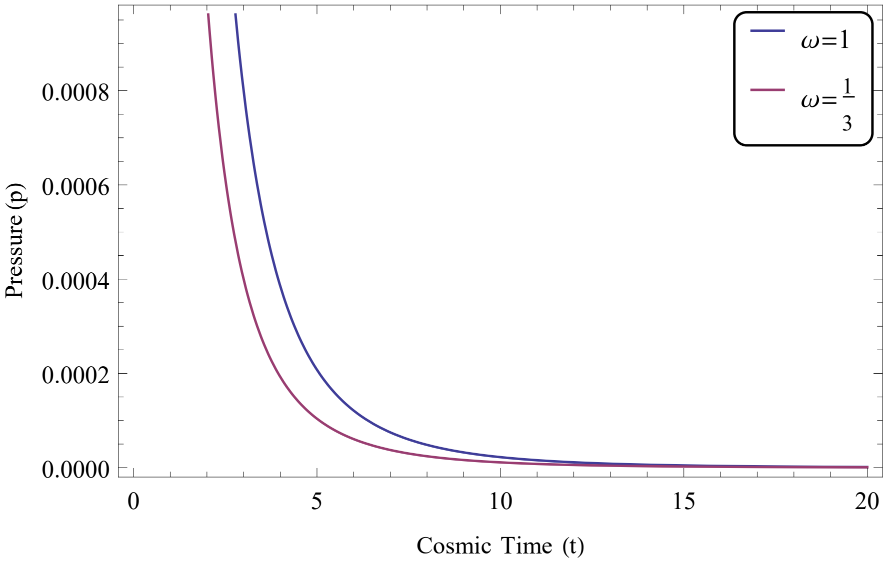

Figure 5.5: The plot of p versus cosmic time t, for $\beta = m = 1, \omega = 0, 1, \frac{1}{3}$.

Eqn. (5.29), represents the expression for the gravitational constant G for the model (5.25). Fig.5.2 shows that the variation of G with respect to cosmic time t for $\beta = 1$ and three different value of $m = 0.5, 1, 1.5$. From this figure, we observed that G is a positive increasing function of time, and tends to infinity as $t \to \infty$, which suggest that the present universe is expanding (Abdel-Rahman (1990)). The model universe shows that the gravitational constant G varies with cosmic time t as suggested by Dirac (Dirac (1937)).

Eqn. (5.32), represents the expression for the cosmological constant Λ for the model (5.25). In Fig. 5.3 we have shown the variation of Λ versus cosmic time t for all three values of $\omega = 0, \omega = 1$ and $\omega = \frac{1}{3}$ corresponding to matter dominated model, zeldovich model and radiating dominated model respectively. From this figure we observed that cosmological constant Λ is infinite as $t \to 0$ and as $t \to \infty$, the cosmological constant Λ converging to a small positive value. Such type of behavior for Λ shows that the expansion will tends to accelerate, which is in good agreement with Type Ia Supernova observations (Perlmutter et al. (1998); Riess et al. (1998); Perlmutter et al. (1999); Bennett et al. (2003); Riess et al. (2004)).

From eqns. (5.30) and (5.31), represents the expressions for energy density ρ and pressure p for the model (5.25). Figs. 5.4 and 5.5 shows that the variation of ρ and p versus cosmic time t for all three types of models of the universe, matter dominated model $\omega = 0$, Zeldovich model $\omega = 1$ and radiating dominated model $\omega = \frac{1}{3}$ respectively. From these figures, it is seen that the energy density ρ and pressure p both are diverges as $t \to 0$ and becomes zero as $t \to \infty$, thus has an initial singularity.

Eqn. (5.37) represents the expression for total energy density Ω. We observed that $\Omega \to 2.33$ as $t \to \infty$. From the present day's observation, the value of total energy density $\Omega \to 1$. But in the proposed model the value of total energy density $\Omega \to 2.33$ as $t \to \infty$ (recently Goyal et al. (2019) found $\Omega \to 3$) . This difference comes out due to use of conharmonically flat space which reduced the Einstein's field equations in to the form given by eqn. (5.8).