CHAPTER 6

Separation Axioms in Neutro-Topological and Anti-Topological Spaces

In this chapter some separation axioms are studied in N-TSs as well as in A-TSs and various hereditary properties that are generally true in topological spaces are observed minutely and various comparisons are made among the various spaces that have been introduced in N-TSs and A-TSs. Many of the hereditary and other relevant properties are found to follow in the N-TS and A-TS with certain exceptions, the reasons for which have been given.

6.1 Separation Axioms in Neutro-Topological Spaces

Definition 6.1.1

A N-TS $(\mathcal{X},\mathcal{T})$ will be called a Nu-T_0 space (T_0^N in short) if for arbitrary elements $m \neq n$ there exists $\mathcal{Q} \in \mathcal{T}$ for which, if $m \in \mathcal{Q}, n \notin \mathcal{Q}$ or, if $n \in \mathcal{Q}$, $m \notin \mathcal{Q}$. In other words, for any set of two unequal points in the space there will exist a N-OS that contains one of the points but not the other.

Proposition 6.1.1

Let a N-TS $(\mathcal{X},\mathcal{T})$ be a T_0^N space. Then for distinct points: $x, y \in \mathcal{X}$, we have: $[\{x\}]^{Nu-cl} \cap [\{y\}]^{Nu-cl} = \emptyset$.

Proof:

Let $(\mathcal{X},\mathcal{T})$ be T_0^N and $x \neq y$ be arbitrary elements in $\mathcal{X}$, then there will be $\wp \in \mathcal{T}$ so that whenever $x \in \wp, y \notin \wp$ or, whenever $y \in \wp$, $x \notin \wp$.

Now, whenever $x \in \wp$, $\{x\} \subseteq \wp$ and whenever $y \notin \wp$, $\{y\} \not\subset \wp$

Again, whenever $\{x\} \subseteq \wp$, $[\{x\}]^{Nu-cl} \subseteq \wp^{Nu-cl}$ by ***proposition 2.3.3 (iii)***

Also, $\{y\} \not\subset \wp \Rightarrow [\{y\}]^{Nu-cl} \not\subset \wp^{Nu-cl}$.

Thus, $[\{x\}]^{Nu-cl} \cap [\{y\}]^{Nu-cl} = \emptyset$.

The neutro-topology part of this chapter has been communicated to an international journal for publication.

Corollary 6.1.1

Let a N-TS $(\mathcal{X},\mathcal{T})$ be a T_0^N space. Then for arbitrary $a \neq b \in \mathcal{X}$, $a \notin [\{b\}]^{Nu-cl}$ and $b \notin [\{a\}]^{Nu-cl}$.

Proposition 6.1.2

For every GTS $(\mathcal{X},\mathcal{T})$, which is T_0, the N-TS $(\mathcal{X},\mathcal{T} \setminus \emptyset)$ is T_0^N.

Proof: By ***theorem 1.6.15***, if $(\mathcal{X},\mathcal{T})$ is a *GTS* then $(\mathcal{X},\mathcal{T} \setminus \emptyset)$ is a *N-TS*.

Proposition 6.1.3

For every GTS $(\mathcal{X},\mathcal{T})$, which is T_0, the N-TS $(\mathcal{X},\mathcal{T} \setminus \mathcal{X})$ is T_0^N.

Proof: By ***theorem 1.6.16***, if $(\mathcal{X},\mathcal{T})$ is a *GTS* then $(\mathcal{X},\mathcal{T} \setminus \mathcal{X})$ is a *N-TS*.

Remark 6.1.1

Propositions 6.1.2 and ***6.1.3*** show that a T_0^N space can be deduced from every T_0 space.

Proposition 6.1.4

If a one-one, onto, and N-O mapping f exist between two N-TSs $(\mathcal{X},\mathcal{T}_x)$ and $(\mathcal{Y},\mathcal{T}_y)$ and if $(\mathcal{X},\mathcal{T}_x)$ is T_0^N then the space $(\mathcal{Y},\mathcal{T}_y)$ is also a T_0^N.

Proof:

When $(\mathcal{X},\mathcal{T}_x)$ is T_0^N and f is a one-one, *N-O* mapping, let us assume two distinct points $y_1 \neq y_2 \in \mathcal{Y}$. Now, since f is onto, there will be members $x_1 \neq x_2 \in \mathcal{X}$ so that $f(x_1) = y_1$ and $f(x_2) = y_2$. Again, since $(\mathcal{X},\mathcal{T}_x)$ is a T_0^N space, there is a $\wp \in \mathcal{T}_x$ which contains one of x_1 or x_2 only and not the other. If $x_1 \in \wp$ then $f(x_1) \in f(\wp) \in \mathcal{T}_y$ since f is a *N-O* map. Thus, $y_1 \in f(\wp) \in \mathcal{T}_y$ which shows that $f(\wp) \in \mathcal{T}_y$ contains y_1 but not y_2 and hence the space $(\mathcal{Y},\mathcal{T}_y)$ is also T_0^N since the points y_1 and y_2 are distinct and moreover arbitrary.

Definition 6.1.2

A N-TS $(\mathcal{X},\mathcal{T})$ will be called as Nu-T_1 (T_1^N in short) if for each arbitrary pair of points $p \neq q$ in $\mathcal{X}$ there exists $\mathcal{P},\mathcal{Q} \in \mathcal{T}$ which satisfy $p \in \mathcal{P} \setminus \mathcal{Q}$ and, $q \in \mathcal{Q} \setminus \mathcal{P}$.

Proposition 6.1.5

If the singleton subsets of a N-TS $(\mathcal{X},\mathcal{T})$ are N-C then the N-TS will be a T_1^N space.

Proof:

For two arbitrary points: x, y in $\mathcal{X}$, if it is assumed that the singleton $\{x\} = \mathcal{A} \subseteq \mathcal{X}$, is a *N-CS*, which means $c\mathcal{A} \in \mathcal{T}$ with $x \notin c\mathcal{A}$ but $y \in c\mathcal{A}$. Analogously, $c(\mathcal{B} = \{y\}) \in \mathcal{T}$ with $y \notin c\mathcal{B}$ but $x \in c\mathcal{B}$. Hence, if we assume: $\mathcal{P} = c\mathcal{B}$ and $\mathcal{Q} = c\mathcal{A}$, then the space $(\mathcal{X}, \mathcal{T})$ will satisfy the condition for being a T_1^N space if the singleton subsets are closed.

Remark 6.1.2

The converse of ***proposition 6.1.5*** is however not true and can be observed from the following example: Let us take the set: $\mathcal{X} = \{1,2,3,4,5\}$, and $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}, \{1,5\}, \{2,3\}, \{3,4\}, \{4,5\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. For any pair of distinct points, the condition for T_1^N is satisfied but none of the singleton subsets of $\mathcal{X}$ are *N-C*. However, as seen in the ***proposition 6.1.5***, when the singletons are closed then the space is a T_1^N space.

Proposition 6.1.6

If a one-one, onto, and N-O mapping f exists between two N-TSs $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$ and if $(\mathcal{X}, \mathcal{T}_x)$ is T_1^N then $(\mathcal{Y}, \mathcal{T}_y)$ is also a T_1^N space.

Proof:

When $(\mathcal{X}, \mathcal{T}_x)$ is T_1^N and f is a one-one and *N-O* mapping, let us assume two distinct points $q_1 \neq q_2 \in \mathcal{Y}$. Now, since f is onto, there will be distinct members $p_1, p_2 \in \mathcal{X}$ that satisfy $f(p_1) = q_1$ and $f(p_2) = q_2$. Again, $(\mathcal{X}, \mathcal{T}_x)$ being a T_1^N space, there are $\mathcal{P}, \mathcal{Q} \in \mathcal{T}_x$ which satisfy $m \in \mathcal{P} \setminus \mathcal{Q}$ and, $n \in \mathcal{Q} \setminus \mathcal{P}$ where m and n are some random points in the space. Further f being a *N-O* map, $f(\mathcal{Q}), f(\mathcal{Q}) \in \mathcal{T}_y$ and as such, we have $q_1 = f(p_1) \in f(\mathcal{P}) \setminus f(\mathcal{Q})$ and $q_2 = f(p_2) \in f(\mathcal{Q}) \setminus f(\mathcal{P})$ thereby showing that $(\mathcal{Y}, \mathcal{T}_y)$ is also a T_1^N space, the points q_1 and q_2 being distinct and arbitrary.

Proposition 6.1.7

Every sub-space of a T_1^N space is also a T_1^N space.

Proof:

Let us assume that the *N-TS* $(\mathcal{X}, \mathcal{T}_x)$ is T_1^N and say, $(\mathcal{Y}, \mathcal{T}_y)$ is a sub-space of the space $(\mathcal{X}, \mathcal{T}_x)$ and say, $y_1 \neq y_2$ are two arbitrary points in $\mathcal{Y}$. Then $\mathcal{Y}$ being a sub-space of $\mathcal{X}$, so y_1, y_2 will be arbitrary points in $\mathcal{X}$ and by virtue of being a T_1^N space there will be two *N-OSs* $\wp_{y_1}, \wp_{y_2}$ in $\mathcal{T}_x$ so that $y_1 \in \wp_{y_1} \setminus \wp_{y_2}$ and $y_2 \in \wp_{y_2} \setminus \wp_{y_1}$. Now, in the sub-space $\mathcal{Y}$ we will have *N-OSs* $\wp_1 = \wp_{y_1} \cap \mathcal{Y}$ and $\wp_2 = \wp_{y_2} \cap \mathcal{Y}$ so that $y_1 \in \wp_1 \setminus \wp_2$ and $y_2 \in \wp_2 \setminus \wp_1$ and as such the sub-space $(\mathcal{Y}, \mathcal{T}_y)$ becomes a T_1^N space.

Proposition 6.1.8

Every T_1^N space is also a T_0^N space.

Proof:

Assume that a *N-TS* $(\mathcal{X}, \mathcal{T})$ is T_1^N, then for arbitrary set of two unequal points p, q there are $\mathcal{P}, \mathcal{Q} \in \mathcal{T}$ that satisfy $p \in \mathcal{P} \cap c\mathcal{Q}$ and, $q \in \mathcal{Q} \cap c\mathcal{P}$ and this means if $p \in \mathcal{P}, q \notin \mathcal{P}$ and if $q \in \mathcal{Q}, p \notin \mathcal{Q}$. Hence $(\mathcal{X}, \mathcal{T})$ is T_0^N.

Proposition 6.1.9

For every T_1 GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is T_1^N.

Proof: By ***theorem 1.6.15***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is a *N-TS*.

Proposition 6.1.10

For every T_1 GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is T_1^N.

Proof: By ***theorem 1.6.16***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is a *N-TS*.

Definition 6.1.3

A N-TS $(\mathcal{X}, \mathcal{T})$ will be called a Nu-T_2 space (T_2^N in short) if for an arbitrary set of two unequal points m and n there exist $\mathcal{P}, \mathcal{Q} \in \mathcal{T}$ satisfying $m \in \mathcal{P}$, $n \in \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Proposition 6.1.11

For every T_2 GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is T_2^N.

Proof: By ***theorem 1.6.15***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is a *N-TS*.

Proposition 6.1.12

For every T_2 GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is T_2^N.

Proof: By ***theorem 1.6.16***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is a *N-TS*.

Proposition 6.1.13

Every sub-space of a T_2^N space is also a T_2^N space.

Proof:

Let us assume that the *N-TS* $(\mathcal{X}, \mathcal{T}_x)$ is T_2^N and let $(\mathcal{Y}, \mathcal{T}_y)$ be a sub-space of $(\mathcal{X}, \mathcal{T}_x)$ and say, $y_1 \neq y_2$ are two random points in $\mathcal{Y}$. Then $\mathcal{Y}$ being a subspace of $\mathcal{X}$, so y_1, y_2

happens to be random points in $\mathcal{X}$ and by virtue of being a T_2^N space there will be two *N-OSs* $\wp_{y_1}, \wp_{y_2}$ in $\mathcal{T}_x$ so that so that $y_1 \in \wp_{y_1}$, $y_2 \in \wp_{y_2}$ with $\wp_{y_1} \cap \wp_{y_2} = \emptyset$. Now, in the sub-space $\mathcal{Y}$ we will have *N-OSs* $\wp_1 = \wp_{y_1} \cap \mathcal{Y}$ and $\wp_2 = \wp_{y_2} \cap \mathcal{Y}$ so that $y_1 \in \wp_1$ and $y_2 \in \wp_2$ and $\wp_1 \cap \wp_2 = \emptyset$. Thus, the sub-space $(\mathcal{Y}, \mathcal{T}_y)$ is also a T_2^N space.

Proposition 6.1.14

Every T_2^N space is also a T_1^N space.

Proof:

Let the *N-TS* $(\mathcal{X}, \mathcal{T})$ be a T_2^N space, then for $x \neq y \in \mathcal{X}$ there exist $\mathcal{L}, \mathcal{M} \in \mathcal{T}$ so that $x \in \mathcal{L}$, $y \in \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M} = \emptyset$. The conditions $\mathcal{L} \cap \mathcal{M} = \emptyset$ and $x \in \mathcal{L}$ results in $y \notin \mathcal{L}$ and further $y \in \mathcal{M}$ with $\mathcal{L} \cap \mathcal{M} = \emptyset$ results in $x \notin \mathcal{M}$ and hence the space $(\mathcal{X}, \mathcal{T}_x)$ which is a T_2^N space is also a T_1^N space.

Proposition 6.1.15

In a T_2^N space, the intersection of all N-C Nu-nhds of any point in the space is necessarily a singleton.

Proof:

Let the *N-TS* $(\mathcal{X}, \mathcal{T})$ be T_2^N, then for $m \neq n \in \mathcal{X}$ there are $\mathcal{E}, \mathcal{F} \in \mathcal{T}$ that satisfy $m \in \mathcal{M}$, $n \in \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \emptyset$. Now, $m \in \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N} = \emptyset \Rightarrow m \in \mathcal{M} \subseteq c(\mathcal{N})$. Thus $c(\mathcal{N})$ is a *N-C* Nu-nhd of the point m and $n \notin c(\mathcal{N})$. Thus, the point n will not belong to the intersection of the *N-C* Nu-nhds of m and since the point n happens to be arbitrary, the intersection in context will only consist of the single point m or the singleton $\{m\}$.

Proposition 6.1.16

If a one-one, onto, N-O and Nu-continuous mapping f exists between two N-TSs $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$ and if $(\mathcal{X}, \mathcal{T}_x)$ is T_2^N then the space $(\mathcal{Y}, \mathcal{T}_y)$ is also T_2^N.

Proof:

Let $(\mathcal{X}, \mathcal{T}_x)$ be T_2^N and f be a one-one and *N-O* mapping of $(\mathcal{X}, \mathcal{T}_x)$ onto $(\mathcal{Y}, \mathcal{T}_y)$, let us assume two distinct points $y_1 \neq y_2 \in \mathcal{Y}$. Now, since f is onto, there will be elements $x_1 \neq x_2 \in \mathcal{X}$ so that $f(x_1) = y_1$ and $f(x_2) = y_2$. Again, since $(\mathcal{X}, \mathcal{T}_x)$ is a T_2^N space, there are $\mathcal{O}_x, \mathcal{O}_y \in \mathcal{T}_x$ so that $x \in \mathcal{O}_x$ and, $y \in \mathcal{O}_y$ and $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$. Also, since f is *N-O* so there exist $f(\mathcal{O}_x), f(\mathcal{O}_y) \in \mathcal{T}_y$ so that $y_1 = f(x_1) \in f(\mathcal{O}_x)$, $y_2 = f(x_2) \in f(\mathcal{O}_y)$

and $f(\mathcal{O}_x) \cap f(\mathcal{O}_y) = f(\mathcal{O}_x \cap \mathcal{O}_y) = f(\emptyset) = \emptyset$, since f is one-one and onto. Hence the space $(\mathcal{Y}, \mathcal{T}_y)$ is T_2^N.

Proposition 6.1.17

If a one-one, onto, and Nu-continuous mapping f exists between two N-TSs $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$ and if $(\mathcal{Y}, \mathcal{T}_y)$ is T_2^N then the space $(\mathcal{X}, \mathcal{T}_x)$ is also T_2^N.

Proof:

Assume $x_1 \neq x_2 \in \mathcal{X}$ then since f is one-one, so $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Suppose that $f(x_1) = y_1$ and $f(x_2) = y_2$ or, $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.

Then for $y_1 \neq y_2 \in \mathcal{Y}$ and since $(\mathcal{Y}, \mathcal{T}_y)$ is a T_2^N space, we have $Q_1, Q_2 \in \mathcal{T}_y$ so that $y_1 \in Q_1$ and $y_2 \in Q_2$ and $Q_1 \cap Q_2 = \emptyset$.

Again, since f is Nu-continuous $f^{-1}(Q_1), f^{-1}(Q_1) \in \mathcal{T}_x$ so that we have:

$y_1 \in Q_1 \Rightarrow f^{-1}(y_1) \in f^{-1}(Q_1)$ which in turn implies $x_1 \in f^{-1}(Q_1)$. Similarly, we have: $y_2 \in Q_2 \Rightarrow f^{-1}(y_2) \in f^{-1}(Q_2)$ which in turn implies $x_2 \in f^{-1}(Q_2)$ and moreover, we have: $f^{-1}(Q_1) \cap f^{-1}(Q_2) = f^{-1}(Q_1 \cap Q_2) = f^{-1}(\emptyset) = \emptyset$.

Thus, for two arbitrary points $x_1 \neq x_2 \in \mathcal{X}$, we have $f^{-1}(Q_1), f^{-1}(Q_1) \in \mathcal{T}_x$ so that $x_1 \in f^{-1}(Q_1)$ and $x_2 \in f^{-1}(Q_2)$ and $f^{-1}(Q_1) \cap f^{-1}(Q_2) = \emptyset$.

Hence the space $(\mathcal{X}, \mathcal{T}_x)$ is also T_2^N.

Definition 6.1.4

A N-TS $(\mathcal{X}, \mathcal{T})$ will be called a Nu-regular space if corresponding to any N-CS $\mathcal{C}$ and $x \notin \mathcal{C}$ there are $\mathcal{O}_c, \mathcal{O}_x \in \mathcal{T}$ so that: $\mathcal{C} \subseteq \mathcal{O}_c$, $x \in \mathcal{O}_x$ and $\mathcal{O}_c \cap \mathcal{O}_x = \emptyset$. If the Nu-regular N-TS $(\mathcal{X}, \mathcal{T})$ is also T_1^N then this N-TS is called a Nu-T_3 space (T_3^N in short). That is, a T_3^N space is a Nu-regular space satisfying the conditions for a T_1^N space.

Proposition 6.1.18

For every regular GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is Nu-regular.

Proof: By ***theorem 1.6.15***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is a *N-TS*.

Proposition 6.1.19

For every regular GTS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is Nu-regular.

Proof: By ***theorem 1.6.16***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is a *N-TS*.

Proposition 6.1.20

For a Nu-regular N-TS $(\mathcal{X}, \mathcal{T})$*, for arbitrary* $x \in \mathcal{X}$ *and random Nu-nhd* $\mathcal{N}$ *of* x*, there will be a Nu-nhd* $\mathcal{Q}$ *of* x *so that* $\mathcal{Q}^{Nu-cl} \subseteq \mathcal{N}$*.*

Proof:

Let the space be Nu-regular and assume $\mathcal{N}$ to be a Nu-nhd of x, then there will be a *N-OS* $\mathcal{O}$ so that $x \in \mathcal{O} \subseteq \mathcal{N}$. Now, $c\mathcal{O}$ is *N-CS* and $x \notin c\mathcal{O}$ so by Nu-regularity of the space, we have: $\mathcal{P}, \mathcal{Q} \in \mathcal{T}$ that satisfies $c\mathcal{O} \subseteq \mathcal{P}$, $x \in \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$ that leads to the fact that $\mathcal{Q} \subseteq c\mathcal{P}$.

Also, $\mathcal{Q} \subseteq c\mathcal{P} \Rightarrow \mathcal{Q}^{Nu-cl} \subseteq (c\mathcal{P})^{Nu-cl}$, *by **proposition 2.3.3 (iii)***

$\Rightarrow \mathcal{Q}^{Nu-cl} \subseteq c\mathcal{P}$, since $c\mathcal{P}$ is *N-C* and by ***proposition 2.3.2***.

Also, $c\mathcal{O} \subseteq \mathcal{P} \Rightarrow c\mathcal{P} \subseteq \mathcal{O} \subseteq \mathcal{N}$ and thus: $\mathcal{Q}^{Nu-cl} \subseteq \mathcal{N}$.

Remark 6.1.3

The converse of ***proposition 6.1.20*** is not always true in a *N-TS* as it would be in a *GTS*. This is because, if we assume the condition to be true in the converse part and assume $\mathcal{C}$ to be some *N-CS* so that $x \notin \mathcal{C}$ then $x \in c\mathcal{C}$, with $c\mathcal{C}$ being a *N-OS* and so by the assumed condition there will exist a *N-OS* $\mathcal{O}$ so that $x \in \mathcal{O}$ and $\mathcal{O}^{Nu-cl} \subseteq c\mathcal{C}$ which gives $\mathcal{C} \subseteq c(\mathcal{O}^{Nu-cl})$. But, since in a *N-TS,* $\mathcal{O}^{Nu-cl}$ will not be always *N-CS* [***remark 2.3.1*** and ***remark 2.3.2***]. Thus, $c(\mathcal{O}^{Nu-cl})$ is not always a *N-OS* and because of this the Nu-regularity of the space fails in a *N-TS.*

Proposition 6.1.21

Every T_3^N *space is also a* T_2^N *space.*

Proof:

Let the *N-TS* $(\mathcal{X}, \mathcal{T})$ be T_3^N, then it is both T_1^N and Nu-regular. Thus, for $x_1 \neq x_2 \in \mathcal{X}$, by virtue of being T_1^N there are *N-OSs* $\mathcal{O}_1$ and $\mathcal{O}_2$ so that $x_1 \in \mathcal{O}_1 \setminus \mathcal{O}_2$ and $x_2 \in \mathcal{O}_2 \setminus \mathcal{O}_1$. Now, $x_1 \in \mathcal{O}_1$ means $x_1 \notin c(\mathcal{O}_1)$ and $c(\mathcal{O}_1)$ is a *N-CS* and hence by virtue of being Nu-regular there will be *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ that satisfy $x_1 \in \mathcal{P}$, $c(\mathcal{O}_1) \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Now, $x_2 \in \mathcal{O}_2 \setminus \mathcal{O}_1 \Rightarrow x_2 \notin \mathcal{O}_1 \Rightarrow x_2 \in c(\mathcal{O}_1) \subseteq \mathcal{Q} \Rightarrow x_2 \in \mathcal{Q}$.

Thus, for arbitrary $x_1 \neq x_2 \in \mathcal{X}$, we have *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ satisfying $x_1 \in \mathcal{P}$, $x_2 \in \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Hence, the space $(\mathcal{X}, \mathcal{T})$ which is T_3^N, is also T_2^N.

Proposition 6.1.22

If a one-one, onto, N-O and weakly Nu-continuous mapping f exists between two N-TSs $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$ and if $(\mathcal{X}, \mathcal{T}_x)$ is Nu-regular then the space $(\mathcal{Y}, \mathcal{T}_y)$ is also Nu-regular.

Proof:

We assume $\mathcal{C}$ to be *N-C* with respect to $\mathcal{T}_y$ and let q to be a point in $\mathcal{Y}$ so that $q \notin \mathcal{C}$. Now, since f is one-one and onto, $\exists\, p \in \mathcal{X}$ so that $f(p) = q \Leftrightarrow f^{-1}(q) = p$. Moreover, since f is weakly Nu-continuous, by ***proposition 5.1.4***, $f^{-1}(\mathcal{C})$ is *N-C* with respect to $\mathcal{T}_x$.

Also, $q \notin \mathcal{C} \Rightarrow f^{-1}(q) \notin f^{-1}(\mathcal{C}) \Rightarrow p \notin f^{-1}(\mathcal{C})$.

Thus, $f^{-1}(\mathcal{C})$ is *N-C* in $\mathcal{X}$ and $p \in \mathcal{X}$ such that $p \notin f^{-1}(\mathcal{C})$.

Hence, by the Nu-regularity of the space $\mathcal{X}$, we have *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ that satisfy $p \in \mathcal{P}$, $f^{-1}(\mathcal{C}) \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Now, $p \in \mathcal{P} \Rightarrow f(p) \in f(\mathcal{P}) \Rightarrow q \in f(\mathcal{P})$

And $f^{-1}(\mathcal{C}) \subseteq \mathcal{Q} \Rightarrow f(f^{-1}(\mathcal{C})) \subseteq f(\mathcal{Q}) \Rightarrow \mathcal{C} \subseteq f(\mathcal{Q})$

And $\mathcal{P} \cap \mathcal{Q} = \emptyset \Rightarrow f(\mathcal{P} \cap \mathcal{Q}) = f(\emptyset) \Rightarrow f(\mathcal{P}) \cap f(\mathcal{Q}) = \emptyset$, since f is one-one.

Also f being *N-O* so $f(\mathcal{P})$ and $f(\mathcal{Q})$ are *N-O* with respect to $\mathcal{T}_y$. Thus, for an arbitrary member y in $\mathcal{Y}$ and a *N-CS* $\mathcal{C}$ with respect to $\mathcal{T}_y$ so that $y \notin \mathcal{C}$, we have *N-OSs* $f(\mathcal{P})$ and $f(\mathcal{Q})$ satisfying $y \in f(\mathcal{P})$, $\mathcal{C} \subseteq f(\mathcal{Q})$ and $f(\mathcal{P}) \cap f(\mathcal{Q}) = \emptyset$ thereby showing that the space $(\mathcal{Y}, \mathcal{T}_y)$ is Nu-regular.

Proposition 6.1.23

Every sub-space $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ of a Nu-regular space $(\mathcal{X}, \mathcal{T}_x)$, is Nu-regular.

Proof:

Let us assume that $\mathcal{F}$ be an arbitrary $\mathcal{T}_{\mathcal{A}}$-*N-CS* and y be an arbitrary point in $\mathcal{A}$ so that $y \notin \mathcal{F}$. Now, by ***proposition 2.5.1 (ii)*** we have $\mathcal{F}_{\mathcal{Y}}^{Nu-cl} = \mathcal{F}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{A}$ where $\mathcal{F}_{\mathcal{X}}^{Nu-cl}$ is the *N-C* of $\mathcal{F}$ in the space $(\mathcal{X}, \mathcal{T}_x)$. Also $\mathcal{F}$ being *N-C* with respect to $\mathcal{T}_{\mathcal{A}}$ we have $\mathcal{F}_{\mathcal{A}}^{Nu-cl} = \mathcal{F}$ and so we have: $\mathcal{F} = \mathcal{F}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{A}$ .... *(1)*

Now, $y \notin \mathcal{F} \Rightarrow y \notin \mathcal{F}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{A} \Rightarrow y \notin \mathcal{F}_{\mathcal{X}}^{Nu-cl}$ as $y \in \mathcal{A}$.

Now, by ***proposition 2.5.1 (i)*** and *(1)* we have $\mathcal{F}_{\mathcal{X}}^{Nu-cl}$ to be *N-C* with respect to $\mathcal{T}_x$ and we have a point $y \notin \mathcal{F}_{\mathcal{X}}^{Nu-cl}$ and so by the Nu-regularity of the space $(\mathcal{X}, \mathcal{T}_x)$, we have

N-OSs $\mathcal{G}$ and $\mathcal{H}$ in $\mathcal{T}_x$ so that $y \in \mathcal{G}$, $\mathcal{F}_{\mathcal{X}}^{Nu-cl} \subseteq \mathcal{H}$ and $\mathcal{G} \cap \mathcal{H} = \emptyset$. Now, $y \in \mathcal{A}$ with $y \in \mathcal{G} \Rightarrow y \in \mathcal{G} \cap \mathcal{A}$ and $\mathcal{F}_{\mathcal{X}}^{Nu-cl} \subseteq \mathcal{H} \Rightarrow \mathcal{F}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{A} \subseteq \mathcal{H} \cap \mathcal{A} \Rightarrow \mathcal{F} \subseteq \mathcal{H} \cap \mathcal{A}$, from *(1)*. Also, $(\mathcal{G} \cap \mathcal{A}) \cap (\mathcal{H} \cap \mathcal{A}) = (\mathcal{G} \cap \mathcal{H}) \cap \mathcal{A} = \emptyset \cap \mathcal{A} = \emptyset$. If we put $\mathcal{G} \cap \mathcal{A} = \mathcal{P}$ and $\mathcal{H} \cap \mathcal{A} = \mathcal{Q}$, then $\mathcal{P}$ and $\mathcal{Q}$ are *N-OSs* in $\mathcal{T}_{\mathcal{A}}$ since $\mathcal{G}$ and $\mathcal{H}$ are *N-O* in $\mathcal{T}_x$. Thus, for arbitrary *N-CS* $\mathcal{F}$ in $\mathcal{A}$ and an arbitrary point $y \notin \mathcal{F}$, we have $y \in \mathcal{P}$, $\mathcal{F} \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$ thereby showing that $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ is Nu-regular.

Proposition 6.1.24

A sub-space $(\mathcal{Y}, \mathcal{T}_y)$ *of a* T_3^N *space* $(\mathcal{X}, \mathcal{T}_x)$ *is also* T_3^N.

Proof:

A T_3^N space is a T_1^N space which is Nu-regular. By ***proposition 6.1.7*** a sub-space of a T_1^N is a T_1^N space and by ***proposition 6.1.23*** a sub-space of a Nu-regular space is Nu-regular. Thus, if $(\mathcal{X}, \mathcal{T}_x)$ is T_3^N then it is both T_1^N and Nu-regular, thus by the ***propositions 6.1.7*** and ***6.1.23***, the sub-space $(\mathcal{Y}, \mathcal{T}_y)$ of $(\mathcal{X}, \mathcal{T}_x)$ is also a T_3^N space.

Proposition 6.1.25

For every T_3 *GTS* $(\mathcal{X}, \mathcal{T})$, *the N-TS* $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ *is* T_3^N.

Proof: By ***theorem 1.6.15***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ is a *N-TS*.

Proposition 6.1.26

For every T_3 *GTS* $(\mathcal{X}, \mathcal{T})$, *the N-TS* $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ *is* T_3^N.

Proof: By ***theorem 1.6.16***, if $(\mathcal{X}, \mathcal{T})$ is a *GTS* then $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ is a *N-TS*.

Definition 6.1.5

A N-TS $(\mathcal{X}, \mathcal{T})$ *will be termed a Nu-normal space if corresponding to a pair of disjoint N-CSs* $\mathcal{C}$ *and* $\mathcal{D}$, *there exists* $\mathcal{O}_C, \mathcal{O}_D \in \mathcal{T}$ *so that:* $\mathcal{C} \subseteq \mathcal{O}_C$, $\mathcal{D} \in \mathcal{O}_D$ *and* $\mathcal{O}_C \cap \mathcal{O}_D = \emptyset$. *If the space* $(\mathcal{X}, \mathcal{T})$ *is also* T_1^N *then the space is called a Nu-*T_4 *space (*T_4^N *in short).*

Proposition 6.1.27

Let a N-TS $(\mathcal{X}, \mathcal{T})$ *be Nu-normal. Then for any N-CS* $\mathcal{F}$ *and a N-OS* $\mathcal{G}$ *which contain* $\mathcal{F}$, *there exists a N-OS* $\mathcal{V}$ *so that* $\mathcal{F} \subseteq \mathcal{V}$ *and* $\mathcal{V}^{Nu-cl} \subseteq \mathcal{G}$.

Proof:

Let us first assume that the space $(\mathcal{X}, \mathcal{T})$ be Nu-normal and $\mathcal{F}$ is some *N-CS* and $\mathcal{G}$ is some *N-OS* in $\mathcal{T}$ such that $\mathcal{F} \subset \mathcal{G}$. Then $c\mathcal{G}$ is *N-C* and $\mathcal{F} \cap c\mathcal{G} = \emptyset$. Thus, $\mathcal{F}$ and $c\mathcal{G}$ are disjoint *N-CSs* and hence by the property of Nu-normality of the space there will be two *N-OSs* $\mathcal{U}$ and $\mathcal{V}$ that satisfy $c\mathcal{G} \subseteq \mathcal{U}$, $\mathcal{F} \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Now, $\mathcal{U} \cap \mathcal{V} = \emptyset \Rightarrow \mathcal{V} \subseteq c\mathcal{U}$, with $c\mathcal{U}$ being *N-C.*

Also $\mathcal{V} \subseteq c\mathcal{U} \Rightarrow \mathcal{V}^{Nu-cl} \subseteq (c\mathcal{U})^{Nu-cl} = c\mathcal{U}$, $c\mathcal{U}$ being *N-C.*

Also $c\mathcal{G} \subseteq \mathcal{U} \Rightarrow c\mathcal{U} \subseteq \mathcal{G}$ and hence $\mathcal{V}^{Nu-cl} \subseteq \mathcal{G}$.

Thus, we get $\mathcal{F} \subseteq \mathcal{V}$ and $\mathcal{V}^{Nu-cl} \subseteq \mathcal{G}$.

Proposition 6.1.28

If $(\mathcal{Y}, \mathcal{T}_y)$ is Nu-homomorphic to a Nu-normal N-TS $(\mathcal{X}, \mathcal{T}_x)$, then $(\mathcal{Y}, \mathcal{T}_y)$ is also Nu-normal.

Proof:

We assume $\mathcal{F}$ and $\mathcal{G}$ to be two random disjoint *N-CSs* with respect to $\mathcal{T}_y$ and let ψ be a Nu-homomorphism between $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$. Then ψ is a weakly Nu-continuous map and as such $\psi^{-1}(\mathcal{F})$ and $\psi^{-1}(G)$ are *N-C* with respect to $\mathcal{T}_x$, by ***proposition 5.1.4***.

Also, $\psi^{-1}(\mathcal{F}) \cap \psi^{-1}(\mathcal{G}) = \psi^{-1}(\mathcal{F} \cap \mathcal{G}) = \psi^{-1}(\emptyset) = \emptyset$, since$\psi$is one-one.

Thus, $\psi^{-1}(\mathcal{F})$ and $\psi^{-1}(\mathcal{G})$ are disjoint *N-CSs* with respect to $\mathcal{T}_x$ and since the space $(\mathcal{X}, \mathcal{T}_x)$ is Nu-normal, so there will be *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{T}_x$, so that $\psi^{-1}(\mathcal{F}) \subseteq \mathcal{P}$ and $\psi^{-1}(\mathcal{G}) \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Now, $\psi^{-1}(\mathcal{F}) \subseteq \mathcal{P} \Rightarrow \psi[\psi^{-1}(\mathcal{F})] \subseteq \psi(\mathcal{P}) \Rightarrow \mathcal{F} \subseteq \psi(\mathcal{P})$ and similarly $\mathcal{G} \subseteq \psi(\mathcal{Q})$. Also, ψ being *N-O*, by ***proposition 5.1.17***, the sets $\psi(\mathcal{P})$ and $\psi(\mathcal{Q})$ are *N-O* in $\mathcal{T}_y$ such that $\psi(\mathcal{P}) \cap \psi(\mathcal{Q}) = \psi(\mathcal{P} \cap \mathcal{Q}) = \emptyset$, since ψ is one-one. Thus, if we put $\psi(\mathcal{P}) = \mathcal{M}$ and $\psi(\mathcal{Q}) = \mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are *N-O* in $\mathcal{T}_y$ and $\mathcal{F} \subseteq \mathcal{M}$, $\mathcal{G} \subseteq \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \emptyset$. This leads to the conclusion that $(\mathcal{Y}, \mathcal{T}_y)$ is also Nu-normal.

Proposition 6.1.29

Every T_4^N space is also a T_3^N space.

Proof:

If the *N-TS* $(\mathcal{X}, \mathcal{T})$ is T_4^N, then it is T_1^N and Nu-normal. Thus, it would be sufficient to show that $(\mathcal{X}, \mathcal{T})$ is Nu-regular.

Now, since $\mathcal{X}$ is Nu-normal, so for two arbitrary disjoint *N-CSs* $\mathcal{F}$ and $\mathcal{G}$, there exist *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ satisfying $\mathcal{F} \subseteq \mathcal{P}$, $\mathcal{G} \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Now, if we assume the *N-CS* $\mathcal{F}$ in $\mathcal{X}$, which was chosen arbitrarily and a random point x in $\mathcal{G}$ so that $x \notin \mathcal{F}$ as $\mathcal{F} \cap \mathcal{G} = \emptyset$ then the *N-OSs* $\mathcal{P}$ and $\mathcal{Q}$ that satisfy $x \in \mathcal{Q}, \mathcal{F} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Hence, $(\mathcal{X}, \mathcal{T})$ is also a T_3^N space.

6.2 Separation Axioms in Anti-Topological Spaces

Definition 6.2.1

An A-TS $(\mathcal{X}, \mathcal{T})$ will be an anti-T_0 space (T_0^A in short) if for random elements $x \neq y$ there is a $\mathcal{Q} \in \mathcal{T}$ for which, whenever $x \in \mathcal{Q}, y \notin \mathcal{Q}$ or, whenever $y \in \mathcal{Q}$, $x \notin \mathcal{Q}$. In other words, for any set of two unequal points in the space there will be an anti-open set that enclose one of the points excluding the other.

Proposition 6.2.1

Let an A-TS $(\mathcal{X}, \mathcal{T})$ be an T_0^A space then for arbitrary distinct points x, y in $\mathcal{X}$, $[\{x\}]^{Anti-cl} \cap [\{y\}]^{Anti-cl} = \emptyset$.

Proof:

Assume $\mathcal{X}$ to be T_0^A and $x \neq y$ be arbitrary elements in $\mathcal{X}$, then there will be $\mathcal{L} \in \mathcal{T}$ so that whenever $x \in \mathcal{L}, y \notin \mathcal{L}$ or, whenever $y \in \mathcal{L}, x \notin \mathcal{L}$.

Now, whenever $x \in \mathcal{L}$, $\{x\} \subseteq \mathcal{L}$ and whenever $y \notin \mathcal{L}$, $\{y\} \not\subset \mathcal{L}$

Again, whenever $\{x\} \subseteq \mathcal{L}$, $[\{x\}]^{Anti-cl} \subseteq \mathcal{L}^{Anti-cl}$ by ***proposition 4.3.3 (iii)***

Also, $\{y\} \not\subset \mathcal{L} \Rightarrow [\{y\}]^{Anti-cl} \not\subset \mathcal{L}^{Anti-cl}$. Thus, $[\{x\}]^{Anti-cl} \cap [\{y\}]^{Anti-cl} = \emptyset$.

Corollary 6.2.1

Let an A-TS $(\mathcal{X}, \mathcal{T})$ be an T_0^A space. Then for arbitrary distinct points p, q in $\mathcal{X}$, $p \notin [\{q\}]^{Anti-cl}$ and $q \notin [\{p\}]^{Anti-cl}$.

Proposition 6.2.2

For every T_0^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is T_0^N.

Proof: By ***theorem 1.6.18,*** if $(\mathcal{X}, \mathcal{T})$ is an *A-TS*, then $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is a *N-TS*.

Proposition 6.2.3

For every T_0^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is T_0^N.

Proof: By ***theorem 1.6.19,*** if $(\mathcal{X}, \mathcal{T})$ is an *A-TS*, then $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is a *N-TS*

Remark 6.2.1

Propositions 6.2.2 and ***6.2.3*** shows that a T_0^N space can be obtained from every T_0^A space. And this follows from ***remark 1.6.11*** of ***chapter 1***.

Proposition 6.2.4

If a one-one, onto, and A-O mapping f exists between two A-TSs $(\mathcal{X}, \mathcal{T}_x)$ and $(\mathcal{Y}, \mathcal{T}_y)$ and if $(\mathcal{X}, \mathcal{T}_x)$ is T_0^A then the space $(\mathcal{Y}, \mathcal{T}_y)$ is also T_0^A.

Proof:

When $(\mathcal{X}, \mathcal{T}_x)$ is T_0^A and f is a one-one *A-O* mapping, let us assume two distinct points $y_1 \neq y_2 \in \mathcal{Y}$. Now, since f is onto, there will be elements $x_1 \neq x_2 \in \mathcal{X}$ so that $f(x_1) = y_1$ and $f(x_2) = y_2$. Again, since $(\mathcal{X}, \mathcal{T}_x)$ is an T_0^A space, there is a $\mathcal{R} \in \mathcal{T}_x$ which contains one of x_1 or x_2 only and not the other. If $x_1 \in \mathcal{R}$ then $f(x_1) \in f(\mathcal{P}) \in \mathcal{T}_y$ since f is *A-O*. Thus, $y_1 \in f(\mathcal{R}) \in \mathcal{T}_y$ thereby meaning that $f(\mathcal{R}) \in \mathcal{T}_y$ contains y_1 but not y_2 and hence the space $(\mathcal{Y}, \mathcal{T}_y)$ is also T_0^A since the points y_1 and y_2 are arbitrary.

Definition 6.2.2

An A-TS $(\mathcal{X}, \mathcal{T})$ will be termed as anti-T_1 (T_1^A in short) if for each arbitrary pair of points $p \neq q$ in $\mathcal{X}$ there exist $\mathcal{K}, \mathcal{L} \in \mathcal{T}$ satisfying $p \in \mathcal{K} \setminus \mathcal{L}$ and, $q \in \mathcal{L} \setminus \mathcal{K}$.

Proposition 6.2.5

If the singleton subsets of an A-TS $(\mathcal{X}, \mathcal{T})$ are A-C then the A-TS will be a T_1^A space.

Proof:

For two random points: p, q in $\mathcal{X}$, if we first assume the singleton $\{p\} = \mathcal{P} \subseteq \mathcal{X}$ is *A-C*, then $c\mathcal{P} \in \mathcal{T}$ with $p \notin c\mathcal{P}$ but $q \in c\mathcal{P}$. Analogously, $c(\{q\} = \mathcal{Q}) \in \mathcal{T}$ with $q \notin c\mathcal{Q}$ but $p \in c\mathcal{Q}$. Hence, if we assume: $\mathcal{M} = c\mathcal{Q}$ and $\mathcal{N} = c\mathcal{P}$, then the space $(\mathcal{X}, \mathcal{T})$ will satisfy the condition for being a T_1^A space if the singleton subsets are *A-C*.

Remark 6.2.2

The converse of ***proposition 6.2.5*** is however not true and can be observed from the following example: Let us assume $\mathcal{X} = \{1,2,3,4,5\}$, and the *A-T* $\mathcal{T} = \{\emptyset, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}$, then $(\mathcal{X}, \mathcal{T})$ is an *A-TS*. For any pair of distinct points, the condition for T_1^A is satisfied but none of the singleton subsets of $\mathcal{X}$ are *A-C*. However, as seen in the ***proposition 6.2.5***, when the singletons are closed then the space is T_1^A.

Proposition 6.2.6

If a one-one, onto, and A-O mapping f *exists between two A-TSs* $(\mathcal{X}, \mathcal{T}_x)$ *and* $(\mathcal{Y}, \mathcal{T}_y)$ *and if* $(\mathcal{X}, \mathcal{T}_x)$ *is* T_1^A *then* $(\mathcal{Y}, \mathcal{T}_y)$ *is also* T_1^A.

Proof:

Let $(\mathcal{X}, \mathcal{T}_x)$ be T_1^A and f be a one-one, onto and *A-O* mapping. Let us assume two distinct points $q_1, q_2 \in \mathcal{Y}$. Now, since f is onto, there will be elements $p_1 \neq p_2 \in \mathcal{X}$ satisfying $f(p_1) = q_1$ and $f(p_2) = q_2$. Again, $(\mathcal{X}, \mathcal{T}_x)$ being a T_1^A space, there exist $\mathcal{P}, \mathcal{Q} \in \mathcal{T}_x$ satisfying $p \in \mathcal{P} \setminus \mathcal{Q}$ and, $q \in \mathcal{Q} \setminus \mathcal{P}$ with arbitrary points p, q in $\mathcal{X}$. Further, f being *A-O* $f(\mathcal{Q}), f(\mathcal{Q}) \in \mathcal{T}_y$ and as such, we have $q_1 = f(p_1) \in f(\mathcal{P}) \setminus f(\mathcal{Q})$ and $q_2 = f(p_2) \in f(\mathcal{Q}) \setminus f(\mathcal{P})$ thereby showing that $(\mathcal{Y}, \mathcal{T}_y)$ is also a T_1^A space, the points q_1 and q_2 being distinct and arbitrary.

Proposition 6.2.7

Every sub-space of an T_1^A *space is an* T_1^A *space.*

Proof:

Let us assume that the *A-TS* $(\mathcal{X}, \mathcal{T}_x)$ be T_1^A and let $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ be a sub-space of the space $(\mathcal{X}, \mathcal{T}_x)$. Let $y_1 \neq y_2$ are two arbitrary points in $\mathcal{A}$. Since $\mathcal{A}$ is a sub-space of $\mathcal{X}$, so y_1, y_2 will be arbitrary points in $\mathcal{X}$ and by virtue of being a T_1^A space there will be two *A-OSs* $\mathcal{Q}_{y_1}, \mathcal{Q}_{y_2}$ in $\mathcal{T}_x$ so that so that $y_1 \in \mathcal{Q}_{y_1} \setminus \mathcal{Q}_{y_2}$ and $y_2 \in \mathcal{Q}_{y_2} \setminus \mathcal{Q}_{y_1}$. Now, in the sub-space $\mathcal{A}$ we will have *A-OSs* $\mathcal{Q}_1 = \mathcal{Q}_{y_1} \cap \mathcal{A}$ and $\mathcal{Q}_2 = \mathcal{Q}_{y_2} \cap \mathcal{A}$ so that $y_1 \in \mathcal{Q}_1 \setminus \mathcal{Q}_2$ and $y_2 \in \mathcal{Q}_2 \setminus \mathcal{Q}_1$ and as such the sub-space $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ becomes a T_1^A space.

Proposition 6.2.8

Every T_1^A *space is also an* T_0^A *space.*

Proof:

When $(\mathcal{X}, \mathcal{T})$ is T_1^A, then for arbitrary set of unequal points p, q there exist $\mathcal{K}, \mathcal{L} \in \mathcal{T}$ that satisfy $p \in \mathcal{K} \cap c\mathcal{L}$ and, $q \in \mathcal{L} \cap c\mathcal{K}$ which means that whenever $p \in \mathcal{K}, q \notin \mathcal{K}$ and whenever $q \in \mathcal{L}, p \notin \mathcal{L}$. Hence $(\mathcal{X}, \mathcal{T})$ is T_0^A.

Proposition 6.2.9

For every T_1^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is T_1^N.

Proof: By ***theorem 1.6.18***, if $(\mathcal{X}, \mathcal{T})$ is an *A-TS* then $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is a *N-TS*.

Proposition 6.2.10

For every T_1^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is T_1^N.

Proof: By ***theorem 1.6.19***, if $(\mathcal{X}, \mathcal{T})$ is an *A-TS* then $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is a *N-TS*.

Definition 6.2.3

An A-TS $(\mathcal{X}, \mathcal{T})$ will be called an anti-T_2 space (T_2^A in short) whenever for arbitrary pair of unequal points p and q in $\mathcal{X}$ there exist $\mathcal{K}, \mathcal{L} \in \mathcal{T}$ such that $p \in \mathcal{K}$, $q \in \mathcal{L}$ and $\mathcal{K} \cap \mathcal{L} = \emptyset$.

Proposition 6.2.11

For every T_2^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is T_2^N.

Proof: By ***theorem 1.6.18***.

Proposition 6.2.12

For every T_2^A ATS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is T_2^N.

Proof: By ***theorem 1.6.19***.

Proposition 6.2.13

Every sub-space of an T_2^A space is an T_2^A space.

Proof:

Let us assume that the *A-TS* $(\mathcal{X}, \mathcal{T}_x)$ is T_2^A and let $(\mathcal{A}, \mathcal{T}_\mathcal{A})$ is a sub-space of the space $(\mathcal{X}, \mathcal{T}_x)$. Let $y_1 \neq y_2$ are two arbitrary points in $\mathcal{A}$. Then $\mathcal{A}$ being a subspace of $\mathcal{X}$, so y_1, y_2 will be arbitrary points in $\mathcal{X}$ also and by virtue of being a T_2^A space there will be two *A-OSs* $\mathcal{P}_{y_1}, \mathcal{P}_{y_2}$ in $\mathcal{T}_x$ so that $y_1 \in \mathcal{P}_{y_1}$, $y_2 \in \mathcal{P}_{y_2}$ with $\mathcal{P}_{y_1} \cap \mathcal{P}_{y_2} = \emptyset$. Now, in the sub-space $\mathcal{A}$ we will have *A-OSs* $\mathcal{P}_1 = \mathcal{P}_{y_1} \cap \mathcal{A}$ and $\mathcal{P}_2 = \mathcal{P}_{y_2} \cap \mathcal{A}$ so that $y_1 \in \mathcal{P}_1$ and $y_2 \in \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. Thus, the sub-space $(\mathcal{A}, \mathcal{T}_\mathcal{A})$ is an T_2^A space.

Proposition 6.2.14

Every T_2^A space is an T_1^A space.

Proof:

Let the *A-TS* $(\mathcal{X},\mathcal{T})$ be T_2^A, then if $p \neq q \in \mathcal{X}$ there exist $\mathcal{Q},\mathcal{R} \in \mathcal{T}$ so that $p \in \mathcal{Q}$, $q \in \mathcal{R}$ and $\mathcal{Q} \cap \mathcal{R} = \emptyset$. The conditions $\mathcal{Q} \cap \mathcal{R} = \emptyset$ and $p \in \mathcal{Q}$ results in $q \notin \mathcal{Q}$ and further $q \in \mathcal{R}$ with $\mathcal{Q} \cap \mathcal{R} = \emptyset$ results in $p \notin \mathcal{R}$ and hence the space $(\mathcal{X},\mathcal{T})$ is also a T_1^A space.

Proposition 6.2.15

If a one-one, onto, A-O and anti-continuous mapping f exists between two A-TSs $(\mathcal{X},\mathcal{T}_x)$ and $(\mathcal{Y},\mathcal{T}_y)$ and if $(\mathcal{X},\mathcal{T}_x)$ is T_2^A then the space $(\mathcal{Y},\mathcal{T}_y)$ is also T_2^A.

Proof:

Let $(\mathcal{X},\mathcal{T}_x)$ be T_2^A and f be a one-one and *A-O* mapping of $(\mathcal{X},\mathcal{T}_x)$ onto $(\mathcal{Y},\mathcal{T}_y)$. Let us assume two distinct points $y_1, y_2 \in \mathcal{Y}$. Now, since f is onto, there will exist elements $x_1 \neq x_2 \in \mathcal{X}$ so that $f(x_1) = y_1$ and $f(x_2) = y_2$. Again, since $(\mathcal{X},\mathcal{T}_x)$ is a T_2^A space, there exist $\mathcal{O}_x, \mathcal{O}_y \in \mathcal{T}_x$ so that $x \in \mathcal{O}_x$ and, $y \in \mathcal{O}_y$ and $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$. Further, f being *A-O* $f(\mathcal{O}_x), f(\mathcal{O}_y) \in \mathcal{T}_y$ and as such, we have $y_1 = f(x_1) \in f(\mathcal{O}_x)$ and $y_2 = f(x_2) \in f(\mathcal{O}_y)$ and $f(\mathcal{O}_x) \cap f(\mathcal{O}_y) = f(\mathcal{O}_x \cap \mathcal{O}_y) = f(\emptyset) = \emptyset$, since f is one-one and onto. Hence the space $(\mathcal{Y},\mathcal{T}_y)$ is T_2^A.

Proposition 6.2.16

If a one-one, onto, and anti-continuous mapping f exists between two A-TSs $(\mathcal{X},\mathcal{T}_x)$ and $(\mathcal{Y},\mathcal{T}_y)$ and if $(\mathcal{Y},\mathcal{T}_y)$ is T_2^A then the space $(\mathcal{X},\mathcal{T}_x)$ is also T_2^A.

Proof:

Assume $x_1 \neq x_2 \in \mathcal{X}$ and since f is one-one, so $x_1 \neq x_2$ means $f(x_1) \neq f(x_2)$. Suppose that $f(x_1) = y_1$ and $f(x_2) = y_2$or, $x_1 = f^{-1}(y_1)$and $x_2 = f^{-1}(y_2)$.

Then $y_1 \neq y_2 \in \mathcal{Y}$ and now since $(\mathcal{Y},\mathcal{T}_y)$ is a T_2^A space, we have $Q_1, Q_2 \in \mathcal{T}_y$ so that $y_1 \in Q_1$ and $y_2 \in Q_2$ and $Q_1 \cap Q_2 = \emptyset$.

Again, since f is anti-continuous $f^{-1}(Q_1)$, $f^{-1}(Q_1) \in \mathcal{T}_x$ so that:

$y_1 \in Q_1 \Rightarrow f^{-1}(y_1) \in f^{-1}(Q_1) \Rightarrow x_1 \in f^{-1}(Q_1)$.

Similarly, we have: $y_2 \in Q_2 \Rightarrow f^{-1}(y_2) \in f^{-1}(Q_2) \Rightarrow x_2 \in f^{-1}(Q_2)$ and moreover, we have: $f^{-1}(Q_1) \cap f^{-1}(Q_2) = f^{-1}(Q_1 \cap Q_2) = f^{-1}(\emptyset) = \emptyset$. Thus, for two arbitrary points $x_1 \neq x_2 \in \mathcal{X}$, we have $f^{-1}(Q_1), f^{-1}(Q_1) \in \mathcal{T}_x$ so that $x_1 \in f^{-1}(Q_1)$ and $x_2 \in f^{-1}(Q_2)$ and $f^{-1}(Q_1) \cap f^{-1}(Q_2) = \emptyset$. Hence the space $(\mathcal{X},\mathcal{T}_x)$ is also T_2^A.

Definition 6.2.4

An A-TS $(\mathcal{X},\mathcal{T})$ *will be called an anti-regular space if corresponding to all A-CS* $\mathcal{C}$ *and* $x \notin \mathcal{C}$ *there exist* $\mathcal{O}_c, \mathcal{O}_x \in \mathcal{T}$ *so that:* $\mathcal{C} \subseteq \mathcal{O}_c$, $x \in \mathcal{O}_x$ *and* $\mathcal{O}_c \cap \mathcal{O}_x = \emptyset$. *If the A-TS* $(\mathcal{X},\mathcal{T})$ *is also* T_1^A *then the A-TS is called an anti-*T_3 *space (*T_3^A *in short).*

Proposition 6.2.17

For every anti-regular A-TS $(\mathcal{X},\mathcal{T})$, *the N-TS* $(\mathcal{X},\mathcal{T} \cup \emptyset)$ *is Nu-regular.*

Proof: By ***theorem 1.6.18***.

Proposition 6.2.18

For every anti-regular A-TS $(\mathcal{X},\mathcal{T})$, *the N-TS* $(\mathcal{X},\mathcal{T} \cup \mathcal{X})$ *is Nu-regular.*

Proof: By ***theorem 1.6.19***.

Proposition 6.2.19

Every sub-space of an anti-regular space is anti-regular.

Proof:

Assume $(\mathcal{A},\mathcal{T}_{\mathcal{A}})$ to be a sub-space of an anti-regular *A-TS* $(\mathcal{X},\mathcal{T})$ and let $\mathcal{P}$ be a random $\mathcal{T}_{\mathcal{A}}$-*A-CS* and y be an arbitrary point in $\mathcal{A}$ so that $y \notin \mathcal{P}$. Now, by ***proposition 4.5.1 (ii)*** we have $\mathcal{P}_{\mathcal{Y}}^{Anti-cl} = \mathcal{P}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{A}$ where $\mathcal{P}_{\mathcal{X}}^{Anti-cl}$ is the anti-closure of $\mathcal{P}$in the space $(\mathcal{X},\mathcal{T}_x)$.

Also $\mathcal{P}$ being *A-C* in $\mathcal{T}_{\mathcal{A}}$, $\mathcal{P}_{\mathcal{A}}^{Anti-cl} = \mathcal{P}$ and so we have: $\mathcal{P} = \mathcal{P}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{A}$ .... *(1)*

Now, $y \notin \mathcal{P} \Rightarrow y \notin \mathcal{P}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{A} \Rightarrow y \notin \mathcal{P}_{\mathcal{X}}^{Anti-cl}$ as $y \in \mathcal{A}$.

Now, by ***proposition 4.5.1 (i)*** and *(1)* we have $\mathcal{P}_{\mathcal{X}}^{Anti-cl}$ to be *A-C* with respect to $\mathcal{T}_x$ and we have a point $y \notin \mathcal{P}_{\mathcal{X}}^{Anti-cl}$ and so by the anti-regularity of the space $(\mathcal{X},\mathcal{T}_x)$, we have *A-OSs* $\mathcal{Q}$ and $\mathcal{R}$ in $\mathcal{T}_x$ so that $y \in \mathcal{Q}, \mathcal{P}_{\mathcal{X}}^{Anti-cl} \subseteq \mathcal{R}$ and $\mathcal{Q} \cap \mathcal{R} = \emptyset$. Now, $y \in \mathcal{A}$ with $y \in \mathcal{Q} \Rightarrow y \in \mathcal{Q} \cap \mathcal{A}$ and $\mathcal{P}_{\mathcal{X}}^{Anti-cl} \subseteq \mathcal{R} \Rightarrow \mathcal{P}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{A} \subseteq \mathcal{R} \cap \mathcal{A} \Rightarrow \mathcal{P} \subseteq \mathcal{R} \cap \mathcal{A}$, from *(1)*. Also, $(\mathcal{Q} \cap \mathcal{A}) \cap (\mathcal{R} \cap \mathcal{A}) = (\mathcal{Q} \cap \mathcal{R}) \cap \mathcal{A} = \emptyset \cap \mathcal{A} = \emptyset$.

If we assign$\mathcal{Q} \cap \mathcal{A} = \mathcal{M}and\mathcal{R} \cap \mathcal{A} = \mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are *A-OSs* in $\mathcal{T}_{\mathcal{A}}$ since $\mathcal{Q}$ and $\mathcal{H}$ are *AO* in $\mathcal{T}_x$. Thus, for random *A-CS* $\mathcal{P}$ in $\mathcal{A}$ and random point$y \notin \mathcal{P}$, satisfying $y \in \mathcal{M}$, $\mathcal{P} \subseteq \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \emptyset$ thereby showing that $(\mathcal{A},\mathcal{T}_{\mathcal{A}})$ is anti-regular.

Proposition 6.2.20

A sub-space $(\mathcal{Y},\mathcal{T}_y)$ *of a* T_3^A *space* $(\mathcal{X},\mathcal{T}_x)$ *is also* T_3^A.

Proof:

An T_3^A space is an T_1^A space and is also anti-regular. By ***proposition 6.2.7*** a sub-space of an T_1^A is an T_1^A space and by ***proposition 6.2.19*** a sub-space of an anti-regular space is anti-regular. Thus, if $(\mathcal{X}, \mathcal{T}_x)$ is T_3^A then it is both T_1^A and anti-regular, so by ***propositions 6.2.7*** and ***6.2.19***, the sub-space $(\mathcal{Y}, \mathcal{T}_y)$ of $(\mathcal{X}, \mathcal{T}_x)$ is also an T_3^A space.

Proposition 6.2.21

Every T_3^A space is also an T_2^A space.

Proof:

If the *A-TS* $(\mathcal{X}, \mathcal{T})$ is T_3^A, then it is T_1^A and anti-regular. For $x_1 \neq x_2 \in \mathcal{X}$ we have by virtue of T_1^A space there exist *A-OSs* $\mathcal{O}_1$ and $\mathcal{O}_2$ so that $x_1 \in \mathcal{O}_1 \setminus \mathcal{O}_2$ and $x_2 \in \mathcal{O}_2 \setminus \mathcal{O}_1$. Now, $x_1 \in \mathcal{O}_1$ means $x_1 \notin c(\mathcal{O}_1)$ and $c(\mathcal{O}_1)$ is a *A-CS* and hence by virtue of anti-regularity there exist *A-OSs* $\mathcal{P}$ and $\mathcal{Q}$ so that $x_1 \in \mathcal{P}$, $c(\mathcal{O}_1) \subseteq \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

Now, $x_2 \in \mathcal{O}_2 \setminus \mathcal{O}_1 \Rightarrow x_2 \notin \mathcal{O}_1 \Rightarrow x_2 \in c(\mathcal{O}_1) \subseteq \mathcal{Q} \Rightarrow x_2 \in \mathcal{Q}$.

Thus, for arbitrary $x_1 \neq x_2 \in \mathcal{X}$, we have *A-OSs* $\mathcal{P}$ and $\mathcal{Q}$ satisfying $x_1 \in \mathcal{P}$, $x_2 \in \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$ which shows that the space $(\mathcal{X}, \mathcal{T})$ is also T_2^A.

Proposition 6.2.22

For every T_3^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is T_3^N.

Proof: By ***theorem 1.6.18***.

Proposition 6.2.23

For every T_3^A A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \mathcal{X})$ is T_3^N.

Proof: By ***theorem 1.6.19***.

Definition 6.2.5

An A-TS $(\mathcal{X}, \mathcal{T})$ will be called an anti-normal space if corresponding to a pair of disjoint A-CSs $\mathcal{C}$ and $\mathcal{D}$, there exist $\mathcal{O}_C, \mathcal{O}_D \in \mathcal{T}$ so that: $\mathcal{C} \subseteq \mathcal{O}_C$, $\mathcal{D} \in \mathcal{O}_D$ and $\mathcal{O}_C \cap \mathcal{O}_D = \emptyset$. If the A-TS is also T_1^A then it is called an anti-T_4 space (T_4^A in short).

Proposition 6.2.24

For every anti-normal A-TS $(\mathcal{X}, \mathcal{T})$, the N-TS $(\mathcal{X}, \mathcal{T} \cup \emptyset)$ is Nu-normal.

Proof: By ***theorem 1.6.18***.

Proposition 6.2.25

For every anti-normal ATS $(\mathcal{X},\mathcal{T})$*, the N-TS* $(\mathcal{X},\mathcal{T}\cup\mathcal{X})$ *is Nu-normal.*

Proof: By ***theorem 1.6.19***.

Proposition 6.2.26

If an A-TS $(\mathcal{X},\mathcal{T})$ *is anti-normal then for arbitrary A-CS* $\mathcal{F}$ *and an A-OS* $\mathcal{G}$ *which contain* $\mathcal{F}$*, there exists an A-OS* $\mathcal{V}$ *so that* $\mathcal{F}\subseteq\mathcal{V}$ *and* $\mathcal{V}^{Anti-cl}\subseteq\mathcal{G}$.

Proof:

Let us first assume that the space $(\mathcal{X},\mathcal{T})$ is anti-normal with $\mathcal{F}$ as some *A-CS* and $\mathcal{G}$ as some *A-OS* in $\mathcal{T}$ so that $\mathcal{F}\subset\mathcal{G}$. Then $c\mathcal{G}$ is *A-CS* and $\mathcal{F}\cap c\mathcal{G}=\emptyset$. Thus, $\mathcal{F}$ and $c\mathcal{G}$ are disjoint *A-CSs* and hence by the property of anti-normality of the space there will be two *A-OSs* $\mathcal{K}$ and $\mathcal{L}$ satisfying $c\mathcal{G}\subseteq\mathcal{K}$, $\mathcal{F}\subseteq\mathcal{L}$ and $\mathcal{K}\cap\mathcal{L}=\emptyset$.

Now, $\mathcal{K}\cap\mathcal{L}=\emptyset\Rightarrow\mathcal{L}\subseteq c\mathcal{K}$, with $c\mathcal{K}$ being *A-CS*.

Also, $\mathcal{L}\subseteq c\mathcal{K}\Rightarrow\mathcal{L}^{Anti-cl}\subseteq(c\mathcal{K})^{Anti-cl}=c\mathcal{K}$, $c\mathcal{K}$ being *A-CS*, ***proposition 4.3.2***.

Also, $c\mathcal{G}\subseteq\mathcal{K}\Rightarrow c\mathcal{K}\subseteq\mathcal{G}$ and hence $\mathcal{L}^{Anti-cl}\subseteq\mathcal{G}$.

Thus, we get $\mathcal{F}\subseteq\mathcal{L}$ and $\mathcal{L}^{Anti-cl}\subseteq\mathcal{G}$.

Proposition 6.2.27

Every T_4^A *space is necessarily a* T_3^A *space.*

Proof:

Let the *A-TS* $(\mathcal{X},\mathcal{T})$ be T_4^A, then it is T_1^A and anti-normal. Thus, it would be sufficient to show that $(\mathcal{X},\mathcal{T})$ is anti-regular. Now, since $\mathcal{X}$ is anti-normal, so for two random disjoint *A-CSs* $\mathcal{F}$ and $\mathcal{G}$, we have *A-OSs* $\mathcal{K}$, $\mathcal{L}$ satisfying $\mathcal{F}\subseteq\mathcal{K}$, $\mathcal{G}\subseteq\mathcal{L}$ and $\mathcal{K}\cap\mathcal{L}=\emptyset$. Now, if we can assume the same *A-CS* $\mathcal{F}$ in $\mathcal{X}$, which was also arbitrarily chosen and a random point x in $\mathcal{G}$ so that $x\notin\mathcal{F}$ as $\mathcal{F}\cap\mathcal{G}=\emptyset$ then we have the *A-OSs* $\mathcal{K}$ and $\mathcal{L}$ that satisfy $x\in\mathcal{L}$ and $\mathcal{F}\subseteq\mathcal{K}$ and $\mathcal{K}\cap\mathcal{L}=\emptyset$ thereby showing that $(\mathcal{X},\mathcal{T})$ is also T_3^A.

Proposition 6.2.28

For every T_4^A *ATS* $(\mathcal{X},\mathcal{T})$*, the N-TS* $(\mathcal{X},\mathcal{T}\cup\emptyset)$ *is* T_4^N.

Proof: By ***theorem 1.6.18***.

Proposition 6.2.29

For every T_4^A *ATS* $(\mathcal{X},\mathcal{T})$*, the N-TS* $(\mathcal{X},\mathcal{T}\cup\mathcal{X})$ *is* T_4^N.

Proof: By ***theorem 1.6.19***.