CHAPTER 7

Multi-Neutro-Topological, Multi-Neutro-Bi-topological and Multi-Anti-Topological Spaces

This chapter will be on the use of multisets in context to N-TS, A-TS and N-BTS. The properties of interior, exterior, closure, and boundary that have been studied in N-TS, A-TS, and N-B-TS will be studied with the use of multi-sets. The definition of Multiset Topological space will be extended to define Multi-Neutro-Topological Space (M-N-TS), Multi-Neutro-Bi-Topological Space (M-N-B-TS), Multi-Anti-Topological Space (M-A-TS). Since a multiset is a set where the occurrence of elements in the set may be more than one and also a multiset is exactly the same as the classical set when the multicity of occurrence of the elements is restricted to unity, so most of the properties that are valid for the neutro-interior, neutro-exterior, neutro-closure, and neutro-boundary in N-TSs will be valid in M-N-TS. Similar will be case in the cases of the M-A-TS and M-N-B-TS.

7.1 Multi-Neutro-Topological Space

Definition 7.1.1

Let $\mathcal{M} \in [\mathcal{X}]^n$ and $\mathcal{T} \subseteq \mathcal{P}^*(\mathcal{M})$. We define a multi-neutro-topology (M-N-T) $\mathcal{T}$ on $\mathcal{X}$ if any of the following three statements is satisfied by $\mathcal{T}$:

- *(i) The m-set $\mathcal{M}$ or the empty set is not in $\mathcal{T}$.*
- *(ii) Union of some members of the collection may not be in $\mathcal{T}$.*
- *(iii) Intersection of some members of the collection may not be in $\mathcal{T}$.*

The space $(\mathcal{M}, \mathcal{T})$ will be called a multi-neutro-topological space (M-N-TS) on $\mathcal{X}$.

Remark 7.1.1

The member of the M-N-TS will be called multi-neutro-open msets or M-N-O msets and the m-complement of such sets will be called multi-neutro-closed msets or M-N-C msets.

The neutro-topology part of this chapter has been communicated to an international journal for publication.

Example 7.1.1

Let $\mathcal{X} = \{a, b, c, d\}, n = 4$ and $\mathcal{M} = \{\frac{3}{a}, \frac{2}{b}, \frac{3}{c}, \frac{4}{d}\}$ be a multiset.

Let $\mathcal{T} = \{\emptyset, \{\frac{2}{a}, \frac{1}{b}\}, \{\frac{3}{a}, \frac{1}{b}\}, \{\frac{1}{a}, \frac{2}{b}\}, \{\frac{2}{a}, \frac{1}{c}\}, \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\}, \{\frac{2}{b}, \frac{2}{c}, \frac{2}{d}\}, \{\frac{2}{c}, \frac{2}{d}\}\}$.

Here: $\emptyset \in \mathcal{T}$ but $\mathcal{M} \notin \mathcal{T}$.

Also, $\{\frac{2}{a}, \frac{1}{b}\} \cup \{\frac{3}{a}, \frac{1}{b}\} = \{\frac{3}{a}, \frac{1}{b}\} \in \mathcal{T}$, but $\{\frac{3}{a}, \frac{1}{b}\} \cup \{\frac{1}{a}, \frac{2}{b}\} = \{\frac{3}{a}, \frac{2}{b}\} \notin \mathcal{T}$

And, $\{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\} \cap \{\frac{1}{a}, \frac{2}{b}\} = \{\frac{1}{a}, \frac{2}{b}\} \in \mathcal{T}$ but $\{\frac{2}{b}, \frac{1}{c}\} \cap \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\} = \{\frac{2}{b}, \frac{1}{c}\} \notin \mathcal{T}$.

Hence, $(\mathcal{M}, \mathcal{T})$ is a *M-N-TS* on $\mathcal{X}$

Definition 7.1.2

Let $(\mathcal{M}, \mathcal{T})$ be a M-N-TS on $\mathcal{X}$ and let $\mathcal{A}$ be a submset of $\mathcal{M}$. Then the multi-neutro-interior (MN-Int, in short) of $\mathcal{A}$ is the union of multi-neutro-open (M-N-O) submsets of $\mathcal{A}$ and is denoted by $\mathcal{A}^{MN-Int}$.

Example 7.1.2

Let $\mathcal{X} = \{a, b, c, d\}$, $n = 4$ and $\mathcal{M} = \{\emptyset, \{\frac{4}{a}, \frac{4}{b}, \frac{3}{c}, \frac{4}{d}\}$. We can define a *M-N-T* on $\mathcal{M}$ as: $\mathcal{T} = \{\emptyset, \{\frac{1}{a}, \frac{2}{b}\}, \{\frac{2}{a}, \frac{2}{b}\}, \{\frac{1}{a}, \frac{2}{b}, \frac{3}{c}\}, \{\frac{2}{a}, \frac{2}{c}\}, \{\frac{2}{b}, \frac{3}{c}, \frac{2}{d}\}\}$. Suppose that $\mathcal{A} = \{\frac{2}{a}, \frac{2}{b}, \frac{2}{c}\}$ then $\mathcal{A}^{MN-Int} = \{\frac{1}{a}, \frac{2}{b}\} \cup \{\frac{2}{a}, \frac{2}{b}\} \cup \{\frac{2}{a}, \frac{2}{c}\} = \{\frac{2}{a}, \frac{2}{b}, \frac{2}{c}\} = \mathcal{A}$.

Remark 7.1.2

The MN-Int of a submset $\mathcal{A}$ is not the largest M-N-O submset contained in $\mathcal{A}$.

Proposition 7.1.1

For any submset $\mathcal{A}$, $\mathcal{A}^{MN-Int} \subseteq \mathcal{A}$. If $\mathcal{A}$ is M-N-O then $\mathcal{A}^{MN-Int} = \mathcal{A}$.

Remark 7.1.3

However, the converse of ***proposition 7.1.1*** need not be true and ***example 7.1.2*** shows it. That is, if $\mathcal{A}^{MN-Int} = \mathcal{A}$, then it is not necessary that $\mathcal{A}$ is *M-N-O*.

Proposition 7.1.2

Let $(\mathcal{M}, \mathcal{T})$ be a M-N-TS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ then the following results holds true:

(i) $\mathcal{X}^{MN-Int} \subseteq \mathcal{X}$, $\emptyset^{MN-Int} = \emptyset$.

(ii) $(\mathcal{A}^{MN-Int})^{MN-Int} = \mathcal{A}^{MN-Int}$.

(iii) *If* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{A}^{MN-Int} \subseteq \mathcal{B}^{MN-Int}$.

(iv) $(\mathcal{A} \cap \mathcal{B})^{MN-Int} \subseteq \mathcal{A}^{MN-Int} \cap \mathcal{B}^{MN-Int}$.

Proof:

(i) By ***proposition 7.1.1***., we have: $\mathcal{X}^{MN-Int} \subseteq \mathcal{X}$.

By ***proposition 7.1.1***., $\emptyset^{MN-Int} \subseteq \emptyset$ and $\emptyset \subseteq \emptyset^{MN-Int}$ and so, $\emptyset^{MN-Int} = \emptyset$.

(ii) Let $\mathcal{A}^{MN-Int} = \mathcal{Q} = \cup \{\mathcal{Q}_i$: each $\mathcal{Q}_i$ is *M-N-O*$\}$

Then$(\mathcal{A}^{MN-Int})^{MN-Int} = (\mathcal{Q})^{MN-Int}$

$= \cup \{\mathcal{Q}_i$: each $\mathcal{Q}_i$ is *M-N-O*$\} = \mathcal{Q} = \mathcal{A}^{MN-Int}$

(iii) We have by ***proposition 7.1.1***, $\mathcal{A}^{MN-Int} \subseteq \mathcal{A} \subseteq \mathcal{B}$ and hence $\mathcal{A}^{MN-Int} \subseteq \mathcal{B}$.

Now, $\mathcal{A}^{MN-Int}$ is an mset which is contained in $\mathcal{B}$ and so it will either be the *MN-Int* of $\mathcal{B}$ or contained in the *MN-Int* of $\mathcal{B}$.

That is, $\mathcal{A}^{MN-Int} = \mathcal{B}^{MN-Int}$ or, $\mathcal{A}^{MN-Int} \subseteq \mathcal{B}^{MN-Int} \subseteq \mathcal{B}$.

In either case, $\mathcal{A}^{MN-Int} \subseteq \mathcal{B}^{MN-Int}$ if $\mathcal{A} \subseteq \mathcal{B}$.

(iv) The fact $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ with *(iii)* will give: $(\mathcal{A} \cap \mathcal{B})^{MN-Int} \subseteq \mathcal{A}^{MN-Int}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ with *(iii)* gives: $(\mathcal{A} \cap \mathcal{B})^{MN-Int} \subset \mathcal{B}^{MN-Int}$ thereby showing that: $(\mathcal{A} \cap \mathcal{B})^{MN-Int} \subseteq \mathcal{A}^{MN-Int} \cap \mathcal{B}^{MN-Int}$

Definition 7.1.3

Let $(\mathcal{M}, \mathcal{T})$ *be a M-N-TS on the set* $\mathcal{X}$ *and* $\mathcal{A} \subseteq \mathcal{M}$, *then the multi-neutro-exterior (MN-Ext, in short) of* $\mathcal{A}$ *is defined as the union of the submsets of* $c\mathcal{A}$ *which are M-N-O and is denoted by* $\mathcal{A}^{MN-Ext}$. That is, $\mathcal{A}^{MN-Ext} = \cup \{\mathcal{S}_i : \mathcal{S}_i \subseteq c\mathcal{A}$ *and each* $\mathcal{S}_i$ *is M-N-O*$\}$. *We define:* $\mathcal{X}^{MN-Ext} = \emptyset$ *and* $\emptyset^{MN-Ext} = \mathcal{X}$.

Proposition 7.1.3

If $(\mathcal{M}, \mathcal{T})$ *is a M-N-TS on the set* $\mathcal{X}$ *and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$, *then:*

(i) $\mathcal{A}^{MN-Ext} \subseteq c\mathcal{A}$

(ii) $\mathcal{A}^{MN-Ext} = (c\mathcal{A})^{MN-Int}$

(iii) $\mathcal{A}^{MN-Ext} = [c(\mathcal{A}^{MN-Ext})]^{MN-Ext}$

(iv) $\mathcal{A}^{MN-Int} = (c\mathcal{A})^{MN-Ext}$

(v) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathcal{A}^{MN-Ext} \supseteq \mathcal{B}^{MN-Ext}$

(vi) $(\mathcal{A}^{MN-Ext})^{MN-Ext} \supseteq \mathcal{A}^{MN-Int}$

(vii) $(\mathcal{A} \cup \mathcal{B})^{MN-Ext} \subseteq \mathcal{A}^{MN-Ext} \cap \mathcal{B}^{MN-Ext}$

(viii) *If* $\mathcal{A}$ *is M-N-C then* $\mathcal{A}^{MN-Ext} = c\mathcal{A}$

(ix) $\mathcal{A}^{MN-Int} \cap \mathcal{A}^{MN-Ext} = \emptyset$

Proof:

(i) $\mathcal{A}^{MN-Ext} = (c\mathcal{A})^{MN-Int} \subseteq c\mathcal{A}$ [by ***proposition 7.1.1***]

(ii) By definition: $\mathcal{A}^{MN-Ext} = \cup\{Q_i : Q_i \subseteq c\mathcal{A}$ and each Q_i is *M-N-O*$\}$

$= (c\mathcal{A})^{MN-Int}$, by definition of *MN-Int*

(iii) We have by *(ii)*:

$[c(\mathcal{A}^{MN-Ext})]^{MN-Ext} = [c(c\mathcal{A})^{MN-Int}]^{MN-Ext}$

$= [c\{c(c\mathcal{A})^{MN-Int}\}]^{MN-Int}$, by *(ii)*

$= [(c\mathcal{A})^{MN-Int}]^{MN-Int}$, $c(c\mathcal{A}) = \mathcal{A}$.

$= (c\mathcal{A})^{MN-Int}$, [by ***proposition 7.1.2 (ii)***]

$= \mathcal{A}^{MN-Ext}$

(iv) We have: $(c\mathcal{A})^{MN-Ext} = \cup\{Q_i : Q_i \subseteq c(c\mathcal{A})$ and each Q_i is *M-N-O*$\}$

$= \cup\{Q_i : Q_i \subseteq \mathcal{A}$ and each Q_i is *M-N-O*$\}$

$= \mathcal{A}^{MN-Int}$

(v) We have $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c\mathcal{B} \subseteq c\mathcal{A}$

$\Rightarrow (c\mathcal{B})^{MN-Int} \subseteq (c\mathcal{A})^{MN-Int}$ [by ***proposition 7.1.2 (iii)***]

$\Rightarrow \mathcal{B}^{MN-Ext} \subseteq \mathcal{A}^{MN-Ext}$

(vi) By *(i)* $\mathcal{A}^{MN-Ext} \subseteq c\mathcal{A}$ and by *(v)* we have:

$(\mathcal{A}^{MN-Ext})^{MN-Ext} \supseteq (c\mathcal{A})^{MN-Ext} = (cc\mathcal{A})^{MN-Int} = \mathcal{A}^{MN-Int}$ and hence $(\mathcal{A}^{MN-Ext})^{MN-Ext} \supseteq \mathcal{A}^{MN-Int}$

(vii) We have by *(ii)*:

$(\mathcal{A} \cup \mathcal{B})^{MN-Ext} = (c(\mathcal{A} \cup \mathcal{B}))^{MN-Int}$

$= (c\mathcal{A} \cap c\mathcal{B})^{MN-Int}$

$\subseteq (c\mathcal{A})^{MN-Int} \cap (c\mathcal{B})^{MN-Int}$, [by ***proposition 7.1.2 (iv)***]

$= \mathcal{A}^{MN-Ext} \cap \mathcal{A}^{MN-Ext}$

Hence, $(\mathcal{A} \cup \mathcal{B})^{MN-Ext} \subseteq \mathcal{A}^{MN-Ext} \cap \mathcal{B}^{MN-Ext}$

(viii) Let $\mathcal{A}$ be *M-N-C*, then $c\mathcal{A}$ is *M-N-O.*

Then by *(ii)* $\mathcal{A}^{MN-Ext} = (c\mathcal{A})^{MN-Int} = c\mathcal{A}$ [by ***proposition 7.1.1***]

(ix) We have: $\mathcal{A}^{MN-Int} \cap \mathcal{A}^{MN-Ext} = \mathcal{A}^{MN-Int} \cap (c\mathcal{A})^{MN-Int}$

$\subseteq (\mathcal{A} \cap c\mathcal{A})^{MN-Int}$

$$= \emptyset^{MN-Int} = \emptyset.$$

Hence, $\mathcal{A}^{MN-Int} \cap \mathcal{A}^{MN-Ext} \subseteq \emptyset \Rightarrow \mathcal{A}^{MN-Int} \cap \mathcal{A}^{MN-Ext} = \emptyset$.

Definition 7.1.4

Let $(\mathcal{M}, \mathcal{T})$ be a M-N-TS and $\mathcal{A} \subseteq \mathcal{M}$, then the multi-neutro-closure (MN-Cl, in short) of $\mathcal{A}$ is defined as the intersection of the MN-closed submsets containing $\mathcal{A}$ and will be denoted by $\mathcal{A}^{MN-Cl}$. Thus, $\mathcal{A}^{MN-Cl} = \cap \{\mathcal{C}: \mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{C}$ is M-N-C$\}$. We define: $\mathcal{X}^{MN-Cl} = \mathcal{X}$ and $\emptyset^{MN-Cl} = \emptyset$.

Proposition 7.1.4

If $\mathcal{A}$ is M-N-C then $\mathcal{A}^{MN-Cl} = \mathcal{A}$.

Remark 7.1.4

The converse of ***proposition 7.1.4*** is not always true and can be seen from the following example. Let $\mathcal{X} = \{a, b, c, d\}, n = 3$ and $\mathcal{M} = \{\emptyset, \{\frac{3}{a}, \frac{3}{b}, \frac{2}{c}, \frac{3}{d}\}$. We define a *M-N-T* on the mset $\mathcal{M}$ as: $\mathcal{T} = \{\emptyset, \{\frac{1}{a}, \frac{2}{b}\}, \{\frac{2}{a}, \frac{2}{b}\}, \{\frac{1}{a}, \frac{2}{b}, \frac{2}{c}\}, \{\frac{2}{a}, \frac{2}{c}\}, \{\frac{2}{b}, \frac{2}{c}, \frac{2}{d}\}\}$. Here the *M-N-C* submsets are: $\mathcal{M}, \left\{\frac{2}{a}, \frac{1}{b}, \frac{2}{c}, \frac{3}{d}\right\}, \left\{\frac{1}{a}, \frac{1}{b}, \frac{2}{c}, \frac{3}{d}\right\}, \left\{\frac{2}{a}, \frac{1}{b}, \frac{3}{d}\right\}, \left\{\frac{1}{a}, \frac{3}{b}, \frac{3}{d}\right\}, \{\frac{3}{a}, \frac{1}{b}, \frac{1}{d}\}$

Suppose that $\mathcal{A} = \{\frac{1}{a}, \frac{2}{c}, \frac{3}{d}\}$ then $\mathcal{A}^{MN-Cl} = \left\{\frac{2}{a}, \frac{1}{b}, \frac{2}{c}, \frac{3}{d}\right\} \cap \left\{\frac{1}{a}, \frac{1}{b}, \frac{2}{c}, \frac{3}{d}\right\} = \{\frac{1}{a}, \frac{2}{c}, \frac{3}{d}\} = \mathcal{A}$.

This shows that the *MN-Cl* of a submset $\mathcal{A}$ is not always the largest *M-N-C* mset that contain $\mathcal{A}$. This also shows that $\mathcal{A}^{MN-Cl} = \mathcal{A}$ but $\mathcal{A}$ is not *M-N-C.*

Proposition 7.1.5

Let $(\mathcal{M}, \mathcal{T})$ be a M-N-TS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$, then the following holds:

- *(i)* $\mathcal{A} \subseteq \mathcal{A}^{MN-Cl}$
- *(ii)* $(\mathcal{A}^{MN-Cl})^{MN-Cl} = \mathcal{A}^{MN-Cl}$
- *(iii)* $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{MN-Cl} \subseteq \mathcal{B}^{MN-Cl}$
- *(iv)* $\mathcal{A}^{MN-Cl} \cup \mathcal{B}^{MN-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MN-Cl}$
- *(v)* $(\mathcal{A} \cap \mathcal{B})^{MN-Cl} \subseteq \mathcal{A}^{MN-Cl} \cap \mathcal{B}^{MN-Cl}$

Proof:

- *(i)* By definition the result is straight.
- *(ii)* We have $\mathcal{A}^{MN-Cl} = \cap \{\mathcal{C}: \mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{C}$ is *M-N-C*$\} = \mathcal{B}$(say). Here $\mathcal{B}$ is the smallest submset containing $\mathcal{A}$.

If $\mathcal{B}$ is *M-N-C,* then $\mathcal{B}^{MN-Cl} = \mathcal{B}$ by ***proposition 7.1.4*** and we have the result. However, if $\mathcal{B}$ is not *M-N-C,* which is possible by ***remark 7.1.4****,* then $\mathcal{B}^{MN-Cl} = \cap \{\mathcal{P}: \mathcal{B} \subseteq \mathcal{P}$ and $\mathcal{P}$ is *M-N-C*$\}$,

$= \cap \{\mathcal{Q}: \mathcal{A} \subseteq \mathcal{Q}$ and $\mathcal{Q}$ is *M-N-C*$\} = \mathcal{B}$

This is because $\mathcal{B}$ is the smallest submset containing $\mathcal{A}$ and there will be no other submsets containing $\mathcal{A}$ other than those that are larger than $\mathcal{B}$ and all of which contains $\mathcal{A}$.

(iii) By *(i)* $\mathcal{A} \subseteq \mathcal{A}^{MN-Cl}$ and $\mathcal{B} \subseteq \mathcal{B}^{MN-Cl}$.

Now, $\mathcal{B}^{MN-Cl} = \cap \{\mathcal{P}: \mathcal{B} \subseteq \mathcal{P};\ \mathcal{P}\ is$ *M-N-C*$\}$

And $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{MN-Cl} = \cap \{\mathcal{Q}: \mathcal{A} \subseteq \mathcal{Q}\} \subseteq \cap \{\mathcal{P}: \mathcal{B} \subseteq \mathcal{P}\} = \mathcal{B}^{MN-Cl}$.

(iv) By *(iii),* $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{A}^{MN-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MN-Cl}$

And $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{B}^{MN-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MN-Cl}$

Hence $\mathcal{A}^{MN-Cl} \cup \mathcal{B}^{MN-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MN-Cl}$.

(v) By *(iii),* $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{MN-Cl} \subseteq \mathcal{A}^{MN-Cl}$

And $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{MN-Cl} \subseteq \mathcal{B}^{MN-Cl}$

Hence, $(\mathcal{A} \cap \mathcal{B})^{MN-Cl} \subseteq \mathcal{A}^{MN-Cl} \cap \mathcal{B}^{MN-Cl}$.

Proposition 7.1.6

Let $(\mathcal{M}, \mathcal{T})$ *be a M-N-TS and* $\mathcal{A} \subseteq \mathcal{M}$, *then the following are relations between the MN-Int and MN-Cl:*

(i) $c(\mathcal{A}^{MN-Int}) = (c\mathcal{A})^{MN-Cl}$

(ii) $(c\mathcal{A})^{MN-Int} = c(\mathcal{A}^{MN-Cl})$

(iii) $\mathcal{A}^{MN-Int} = c((c\mathcal{A})^{MN-Cl})$

(iv) $c((c\mathcal{A})^{MN-Int}) = \mathcal{A}^{MN-Cl}$.

Proof:

(i) $\mathcal{A}^{MN-Int} = \cup\, \mathcal{Q}_i$ so that each $\mathcal{Q}_i$ is *M-N-O* and $\mathcal{Q}_i \subseteq \mathcal{A}$.

Thus, $c(\mathcal{A}^{MN-Int}) = c(\cup\, \mathcal{Q}_i)$ so that $c(\mathcal{Q}_i) \supseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{MN-Int}) = \cap\, (c\mathcal{Q}_i)$ so that each $c\mathcal{O}_i$ is *M-N-C* and $c(\mathcal{A}) \subseteq c(\mathcal{Q}_i)$

Or, $c(\mathcal{A}^{MN-Int}) = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *M-N-C* and $c(\mathcal{A}) \subseteq \mathcal{C}_i$

Or, $c(\mathcal{A}^{MN-Int}) = (c\mathcal{A})^{MN-Cl}$

(ii) $\mathcal{A}^{MN-cl} = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *M-N-C* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Thus, $c(\mathcal{A}^{MN-Cl}) = c(\cap\, \mathcal{C}_i)$ so that $c(\mathcal{A}) \supseteq c(\mathcal{C}_i)$

Or, $c(\mathcal{A}^{MN-Cl}) = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *M-N-O* and $c(\mathcal{C}_i) \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{MN-Cl}) = \cup\, (\mathcal{Q}_i)$ so that each $\mathcal{Q}_i$ is *M-N-O* and $\mathcal{Q}_i \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{MN-Cl}) = (c\mathcal{A})^{MN-Int}$

(iii) $(c\mathcal{A})^{MN-Cl} = \cap\, \mathcal{C}_i$, where each $\mathcal{C}_i$ is *M-N-C* and $c\mathcal{A} \subseteq \mathcal{C}_i$.

So, $c(c\mathcal{A})^{MN-Cl} = c(\cap\, \mathcal{C}_i)$ so that $c(c\mathcal{A}) \supseteq c\mathcal{C}_i$

Or, $c(c\mathcal{A})^{MN-Cl} = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *M-N-O* and $c\mathcal{C}_i \subseteq \mathcal{A}$

Or, $c(c\mathcal{A})^{MN-Cl} = \cup\, (\mathcal{D}_i)$ so that each $\mathcal{D}_i$ is *M-N-O* and $\mathcal{D}_i \subseteq \mathcal{A}$.

Or, $c(c\mathcal{A})^{MN-Cl} = \mathcal{A}^{MN-Int}$.

(iv) $(c\mathcal{A})^{MN-Int} = \cup\, \mathcal{B}_i$ so that each $\mathcal{B}_i$ is *M-N-O* and $\mathcal{B}_i \subseteq c\mathcal{A}$

So, $c((c\mathcal{A})^{MN-Int}) = c(\cup\, \mathcal{B}_i)$ so that each $\mathcal{B}_i$ is *M-N-O* and $\mathcal{B}_i \subseteq c\mathcal{A}$

Or, $c((c\mathcal{A})^{MN-Int}) = \cap\, (c\mathcal{B}_i)$ so that each $c\mathcal{B}_i$ is *M-N-C* and $c\mathcal{B}_i \supseteq c(c\mathcal{A})$

Or, $c((c\mathcal{A})^{MN-Int}) = \cap\, (\mathcal{C}_i)$ so that each $\mathcal{C}_i$ is *M-N-C* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Or, $c((c\mathcal{A})^{MN-Int}) = \mathcal{A}^{MN-Cl}$.

Definition 7.1.5

Let $(\mathcal{M}, \mathcal{T})$ be a M-N-TS and $\mathcal{A}$ is a submset of $\mathcal{M}$, then the multi-neutro-boundary (MN-Bd, in short) of $\mathcal{A}$, denoted by $\mathcal{A}^{MN-Bd}$, is defined as:

$\mathcal{A}^{MN-Bd} = \mathcal{A}^{MN-Cl} \cap (c\mathcal{A})^{MN-Cl}$. *In other words, the MN-Bd of $\mathcal{A}$ consists of all those points that do not belong to the MN-Int or the MN-Ext of $\mathcal{A}$.*

Proposition 7.1.7

Let $\mathcal{A}$ be any submset of a M-N-TS $(\mathcal{M}, \mathcal{T})$then defined over a set $\mathcal{X}$ then:

(i) $\mathcal{A}^{MN-Bd} = c(\mathcal{A}^{MN-Int} \cup \mathcal{A}^{MN-Ext})$

(ii) $\mathcal{A}^{MN-Int} \cup (c\mathcal{A})^{MN-Int} = c(\mathcal{A}^{MN-Bd})$

Proof:

(i) By definition, if $x \in \mathcal{A}^{MN-Bd}$ then $x \notin \mathcal{A}^{MN-Int}$ and $x \notin \mathcal{A}^{MN-Ext}$

$$\Leftrightarrow x \notin \mathcal{A}^{MN-Int} \cup \mathcal{A}^{MN-Ext}$$

$$\Leftrightarrow x \in c(\mathcal{A}^{MN-Int} \cup \mathcal{A}^{MN-Ext})$$

Hence, $\mathcal{A}^{MN-Bd} = c(\mathcal{A}^{MN-Int} \cup \mathcal{A}^{MN-Ext})$

(ii) We have, $c(\mathcal{A}^{MN-Bd}) = c(\mathcal{A}^{MN-Cl} \cap (c\mathcal{A})^{MN-Cl})$

$\Rightarrow c(\mathcal{A}^{MN-Bd}) = c(\mathcal{A}^{MN-Cl}) \cup c(c\mathcal{A})^{MN-Cl})$

$\Rightarrow c(\mathcal{A}^{MN-Bd}) = (c\mathcal{A})^{MN-intI} \cup \mathcal{A}^{MN-Int}$ [by ***proposition 7.1.6 (ii)***, ***(iii)***]

Hence, $\mathcal{A}^{MN-Int} \cup (c\mathcal{A})^{MN-Int} = c(\mathcal{A}^{MN-Bd})$.

7.2 Multi-Neutro-Bi-topological Spaces

Definition 7.2.1

If $(\mathcal{T}_1, \mathcal{M})$, $(\mathcal{T}_2, \mathcal{M})$ are two M-N-TSs defined on the multiset $\mathcal{M}$ which is defined on a universe $\mathcal{X}$, [$\mathcal{T}_1$ and $\mathcal{T}_2$ maybe same or different] then the triplet $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ is defined to be a multi-neutro-bi-topological space (M-N-B-TS, in short).

Remark 7.2.1

The submsets of $\mathcal{M}$ that are included in the *M-N-T* $\mathcal{T}_1$ will be called *M-N-O* with respect to $\mathcal{T}_1$ and those included in the *M-N-T* $\mathcal{T}_2$ will be called *M-N-O* with respect to $\mathcal{T}_2$. In this chapter too, no union or intersection of the submsets of $\mathcal{M}$ will be considered and the submsets corresponding to the *M-N-Ts* will be studied separately. Similar will be the case of the *M-N-C* submsets which are the complements of the *M-N-O* submsets.

Definition 7.2.2

Let $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ be a M-N-B-TS and $\mathcal{A} \subseteq \mathcal{M}$ then the MN-Bi-interior of $\mathcal{A}$ is denoted by $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}$ and is defined as the MN-Int with respect to $\mathcal{T}_1$ of the MN-Int of $\mathcal{A}$ with respect to $\mathcal{T}_2$.

That is: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = (\mathcal{A}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int}$, where $\mathcal{A}_{MN}^{\mathcal{T}_2-int} = \cup\, \{\mathcal{Q}_i$: each $\mathcal{Q}_i$ is $\mathcal{T}_2$-M-N-O and $\mathcal{Q}_i \subseteq \mathcal{A}\}$.

Thus, $(\mathcal{A}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int} = \cup\, \{\mathcal{O}_i$: each $\mathcal{O}_i$ is $\mathcal{T}_1$-M-N-O and $\mathcal{O}_i \subseteq \mathcal{A}_{MN}^{\mathcal{T}_2-int}\}$.

Definition 7.2.3

Let $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ be a M-N-B-TS and $\mathcal{A} \subseteq \mathcal{M}$. If $\mathcal{A}$ is M-N-O with respect to both $\mathcal{T}_1$ and $\mathcal{T}_2$ then such submsets will be called $\mathcal{T}_{12}$-M-N-O and their complements will be called $\mathcal{T}_{12}$-M-N-C.

Proposition 7.2.1

Let $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ be a M-N-B-TS and $\mathcal{A} \subseteq \mathcal{M}$. If $\mathcal{A}$ is $\mathcal{T}_{12}$-M-N-O then $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = \mathcal{A}$.

Remark 7.2.2

The converse of ***proposition 7.2.1*** is not always true. That is, if $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = \mathcal{A}$ then $\mathcal{A}$ is not necessarily $\mathcal{T}_{12}$-*M-N-O* and will be seen from the following example.

Let us consider $\mathcal{X} = \{a, b, c, d\}$, $n = 3$ and $\mathcal{M} = \{\frac{3}{a}, \frac{2}{b}, \frac{3}{c}, \frac{2}{d}\}$, $\mathcal{T}_1 = \{\emptyset, \left\{\frac{1}{a}\right\}, \left\{\frac{2}{a}, \frac{2}{b}\right\}, \left\{\frac{2}{b}, \frac{2}{c}\right\}, \left\{\frac{1}{a}, \frac{2}{c}, \frac{2}{d}\right\}, \{\frac{2}{a}, \frac{2}{b}, \frac{3}{c}\}$, $\mathcal{T}_2 = \{\emptyset, \left\{\frac{2}{a}\right\}, \left\{\frac{2}{a}, \frac{3}{c}\right\}, \left\{\frac{1}{b}, \frac{2}{d}\right\}, \left\{\frac{2}{b}, \frac{2}{c}\right\}, \left\{\frac{1}{a}, \frac{2}{b}, \frac{2}{c}\right\}\}$ and $\mathcal{A} = \{\frac{2}{a}, \frac{2}{b}, \frac{2}{c}\}$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are two *M-N-Ts* on $\mathcal{M}$.

Then $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = \left(\mathcal{A}_{MN}^{\mathcal{T}_2-int}\right)^{\mathcal{A}_{MN}^{\mathcal{T}_1-int}} = \left(\left\{\frac{2}{a}\right\} \cup \left\{\frac{2}{b}, \frac{2}{c}\right\}\right)^{\mathcal{T}_1-Nint} = \left(\left\{\frac{2}{a}, \frac{2}{b}, \frac{2}{c}\right\}\right)^{\mathcal{T}_1-Nint}$

$= \left\{\frac{1}{a}\right\} \cup \left\{\frac{2}{a}, \frac{2}{b}\right\} \cup \left\{\frac{2}{b}, \frac{2}{c}\right\} = \{\frac{2}{a}, \frac{2}{b}, \frac{2}{c}\} = \mathcal{A}$, but $\mathcal{A}$ is not $\mathcal{T}_{12}$-*M-N-O*.

Proposition 7.2.2

For a M-N-B-TS $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ *defined over* $\mathcal{X}$ *if* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$, *then the results that follows are true.*

(i) $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}$

(ii) $\emptyset_{MN}^{\mathcal{T}_{12}-int} = \emptyset;\ \mathcal{M}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{M}$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_{12}-int}$

(iv) $[\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}]_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-int}$

(v) $[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-int} \cap \mathcal{B}_{MN}^{\mathcal{T}_{12}-int}$

Proof*:*

(i) By definition the result follows.

(ii) Since the void set is a subset of all sets, we have $\emptyset \subseteq \emptyset_{MN}^{\mathcal{T}_{12}-int}$and by *(i)* we have*:* $\emptyset_{MN}^{\mathcal{T}_{12}-int} \subseteq \emptyset$ and thus, $\emptyset_{MN}^{\mathcal{T}_{12}-int} = \emptyset$.

By *(i)*, $\mathcal{X}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{X}$.

(iii) We have:$\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = (\mathcal{A}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int}$.

Now, $\mathcal{A}_{MN}^{\mathcal{T}_2-int} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_2-int}$, [since $\mathcal{A} \subseteq B$ and [by ***proposition 7.1.2 (iii)***]

Again since $\mathcal{A}_{MN}^{\mathcal{T}_2-int} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_2-int}$, by ***proposition 7.1.2 (iii)*** we have: $(\mathcal{A}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int} \subseteq (\mathcal{B}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int}$ from which, we get: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_{12}-int}$

(iv) $(\mathcal{A}_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int} = \cup\, \{\mathcal{O}_i$: each $\mathcal{O}_i$ is $\mathcal{T}_1$-*M-N-O* and$\mathcal{O}_i \subseteq \mathcal{A}_{MN}^{\mathcal{T}_2-int}\} = \mathcal{B}$, say, and since $\mathcal{B} \subseteq \mathcal{A}$, so by *(iii)*, we have: $\mathcal{B}_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-int}$ which gives:

$$(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int})_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-int}$$

(v) We have: $[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_{12}-int} = ([\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int}$

Now, $[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_2-int} \cap \mathcal{B}_{MN}^{\mathcal{T}_2-int}$, by ***proposition 7.1.2 (iv)*** and as such by *(iii)* of the same proposition, we have:

$([\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int} \subseteq [\mathcal{A}_{MN}^{\mathcal{T}_2-int} \cap \mathcal{B}_{MN}^{\mathcal{T}_2-int}]_{MN}^{\mathcal{T}_1-int}$ and by ***proposition 7.1.2 (iv)***, applied on the right side again, we get:

$([\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-int})_{MN}^{\mathcal{T}_1-int} \subseteq [\mathcal{A}_{MN}^{\mathcal{T}_2-int}]_{MN}^{\mathcal{T}_1-int} \cap [\mathcal{B}_{MN}^{\mathcal{T}_2-int}]_{MN}^{\mathcal{T}_1-int}$

Or, $[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_{12}-int} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-int} \cap \mathcal{B}_{MN}^{\mathcal{T}_{12}-int}$

Definition 7.2.4

Let $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ be a M-N-B-TS and $\mathcal{A} \subseteq \mathcal{M}$ then the MN-Bi-closure of $\mathcal{A}$ is denoted by $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}$ and is defined as the MN-Cl with respect to $\mathcal{T}_1$ of the MN-Cl of $\mathcal{A}$ with respect to $\mathcal{T}_2$. That is: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = [\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl}$ where $\mathcal{A}_{MN}^{\mathcal{T}_2-cl} = \cup \{\mathcal{C}_i$: each $\mathcal{C}_i$ is $\mathcal{T}_2$-M-N-C and $\mathcal{A} \subseteq \mathcal{C}_i\}$. Thus, $[\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl} = \cup \{\mathcal{C}_i$: each $\mathcal{C}_i$ is $\mathcal{T}_1$-M-N-C and $\mathcal{A}_{MN}^{\mathcal{T}_2-cl} \subseteq \mathcal{C}_i\}$. We define: $\emptyset_{MN}^{\mathcal{T}_{12}-cl} = \emptyset$.

Proposition 7.2.3

For a M-N-B-TS $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ is $\mathcal{T}_{12}$-M-N-C then $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = \mathcal{A}$.

Proof:

*If $\mathcal{A}$ is $\mathcal{T}_{12}$-M-N-C, then by **proposition 7.1.4**, we have:*

$\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = [\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl} = [\mathcal{A}]_{MN}^{\mathcal{T}_1-cl} = \mathcal{A}$.

Remark 7.2.3

The converse of the above proposition is not true. That is, if $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = \mathcal{A}$, then it is not necessary that $\mathcal{A}$ is $\mathcal{T}_{12}$-*M-N-C*. The following example may be taken. Let us assume: $\mathcal{X} = \{p, q, r, s\}$, $n = 3$, $\mathcal{M} = \left\{\frac{3}{p}, \frac{3}{q}, \frac{3}{r}, \frac{3}{s}\right\}$, $\mathcal{T}_1 = \left\{\emptyset, \left\{\frac{3}{p}\right\}, \left\{\frac{3}{q}\right\}, \left\{\frac{3}{r}, \frac{3}{s}\right\}, \left\{\frac{3}{p}, \frac{3}{r}, \frac{3}{s}\right\}\right\}$, $\mathcal{T}_2 = \left\{\emptyset, \left\{\frac{3}{r}\right\}, \left\{\frac{3}{s}\right\}, \left\{\frac{3}{p}, \frac{3}{q}\right\}, \left\{\frac{3}{q}, \frac{3}{s}\right\}, \left\{\frac{3}{q}, \frac{3}{r}, \frac{3}{s}\right\}\right\}$ and let $\mathcal{A} = \{\frac{3}{p}, \frac{3}{q}\}$. Now, the$\mathcal{T}_1$-*M-N-C* submsets are: $\mathcal{M}, \left\{\frac{3}{q}, \frac{3}{r}, \frac{3}{s}\right\}, \left\{\frac{3}{p}, \frac{3}{r}, \frac{3}{s}\right\}, \left\{\frac{3}{p}, \frac{3}{q}\right\}, \{\frac{3}{q}\}$ and the $\mathcal{T}_2$ -*M-N-C* submsets are: $\mathcal{M}, \left\{\frac{3}{p}, \frac{3}{q}, \frac{3}{s}\right\}, \left\{\frac{3}{p}, \frac{3}{q}, \frac{3}{r}\right\}, \left\{\frac{3}{r}, \frac{3}{s}\right\}, \left\{\frac{3}{p}\right\}$. We have: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = [\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl} = [\left\{\frac{3}{p}, \frac{3}{q}, \frac{3}{s}\right\} \cap \left\{\frac{3}{p}, \frac{3}{q}, \frac{3}{r}\right\}]_{MN}^{\mathcal{T}_1-cl} = [\left\{\frac{3}{p}, \frac{3}{q}\right\}]_{MN}^{\mathcal{T}_1-cl} = \left\{\frac{3}{p}, \frac{3}{q}\right\} = \mathcal{A}$. But $\mathcal{A}$ is not *M-N-C* with respect to $\mathcal{T}_2$ and so is not $\mathcal{T}_{12}$-*M-N-C*.

Proposition 7.2.4

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *be a M-N-B-TS and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$*, then the following results hold:*

(i) $\mathcal{A} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}$

(ii) $\mathcal{M}_{MN}^{\mathcal{T}_{12}-cl} = \mathcal{M}$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}$

(iv) $(\mathcal{A} \cap \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl} \subseteq \left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\right) \cap \left(\mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}\right)$

(v) $\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\right) \cup \left(\mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}\right) \subseteq (\mathcal{A} \cup \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl}$

Proof*:*

(i) We have: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = [\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl}$

Now, $\mathcal{A} \subseteq \mathcal{A}_{MN}^{\mathcal{T}_2-cl}$, [by ***proposition 7.1.5 (i)***]

So, $\mathcal{A} \subseteq [\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl} = \mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}$

(ii) By *(i)*, $\mathcal{M} \subseteq \mathcal{M}_{MN}^{\mathcal{T}_{12}-cl}$ and since $\mathcal{M}$ is the universal mset, we have: $\mathcal{M}^{\mathcal{T}_{12}-cl} \subseteq \mathcal{M}$ and thus $\mathcal{M}^{\mathcal{T}_{12}-cl} = \mathcal{M}$.

(iii) Since $\mathcal{A} \subseteq \mathcal{B}$, by ***proposition 7.1.5 (iii)*** we have:

$\mathcal{A}_{MN}^{\mathcal{T}_2-cl} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_2-cl}$and by the same proposition we must again have:

$[\mathcal{A}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl} \subseteq [\mathcal{B}_{MN}^{\mathcal{T}_2-cl}]_{MN}^{\mathcal{T}_1-cl}$

Thus, $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} \subseteq \mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}$.

(iv) We have:$[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_{12}-cl} = ([\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-cl})_{MN}^{\mathcal{T}_1-cl}$

Now, $[\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-cl} \subseteq [\mathcal{A}]_{MN}^{\mathcal{T}_2-cl} \cap [\mathcal{B}]_{MN}^{\mathcal{T}_2-cl}$ [by ***proposition 7.1.5 (v)***]

Thus, $([\mathcal{A} \cap \mathcal{B}]_{MN}^{\mathcal{T}_2-cl})_{MN}^{\mathcal{T}_1-cl} \subseteq ([\mathcal{A}]_{MN}^{\mathcal{T}_2-cl} \cap [\mathcal{B}]_{MN}^{\mathcal{T}_2-cl})_{MN}^{\mathcal{T}_1-cl}$,

[by ***proposition 7.1.5 (iii)***]

$\subseteq ([\mathcal{A}]_{MN}^{\mathcal{T}_2-cl})_{MN}^{\mathcal{T}_1-cl} \cap ([\mathcal{B}]_{MN}^{\mathcal{T}_2-cl})_{MN}^{\mathcal{T}_1-cl}$

[by ***proposition 7.1.5 (v)***]

Thus, $(\mathcal{A} \cap \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl} \subseteq \left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\right) \cap \left(\mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}\right)$

(v) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B}$, so by *(iii)*, we have: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} \subseteq (\mathcal{A} \cup \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl}$ and $\mathcal{B}_{MN}^{\mathcal{T}_{12}-cl} \subseteq (\mathcal{A} \cup \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl}$. Thus, we must have: $\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\right) \cup \left(\mathcal{B}_{MN}^{\mathcal{T}_{12}-cl}\right) \subseteq (\mathcal{A} \cup \mathcal{B})_{MN}^{\mathcal{T}_{12}-cl}$

Proposition 7.2.5

For a M-N-B-TS $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$*, if* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$*, then the following results are true:*

(i) $c(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}) = (c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}$

(ii) $c(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}) = (c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}$

Proof*:*

(i) We have: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = [\mathcal{A}_{MN}^{\mathcal{T}_{2}-int}]_{MN}^{\mathcal{T}_{1}-int}$

$= c\{[c(\mathcal{A}_{MN}^{\mathcal{T}_{2}-int})]_{MN}^{\mathcal{T}_{1}-cl}\}$, [by ***proposition 7.1.6 (iii)***]

$= c\{[(c\mathcal{A})_{MN}^{\mathcal{T}_{2}-cl}]_{MN}^{\mathcal{T}_{1}-cl}\}$, [by ***proposition 7.1.6 (i)***]

$= c\{(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}\}$

Hence, $c\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}\right) = c[c\{(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}\}]$

Thus, $c\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}\right) = (c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}$, since $c[c\mathcal{A}] = \mathcal{A}$.

(ii) We have: $(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int} = [(c\mathcal{A})_{MN}^{\mathcal{T}_{2}-int}]_{MN}^{\mathcal{T}_{1}-int}$

$= c\{[c\{(c\mathcal{A})_{MN}^{\mathcal{T}_{2}-int}\}]_{MN}^{\mathcal{T}_{1}-cl}\}$, [by ***proposition 7.1.6 (iii)***]

$= c\left\{[c\{c\{(cc\mathcal{A})_{MN}^{\mathcal{T}_{2}-cl}\}\}]_{MN}^{\mathcal{T}_{1}-cl}\right\}$,

[by ***proposition 7.1.6 (iii)***]

$= c\left\{[(\mathcal{A})_{MN}^{\mathcal{T}_{2}-cl}]_{MN}^{\mathcal{T}_{1}-cl}\right\}$, *by* $c[c\mathcal{A}] = \mathcal{A}$

$= c(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl})$

Corollary 7.2.1

(i) $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}]$

(ii) $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}]$

Proof*:*

(i) Taking complement on both sides of *(i)* of ***proposition 7.2.5***, we have:

$c[\, c(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int})] = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}]$ which give: $\mathcal{A}_{MN}^{\mathcal{T}_{12}-int} = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}]$

(ii) Taking complements of both sides of *(ii)* of ***proposition 7.2.5***, we get:

$c[c(\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl})] = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}]$, or, $\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} = c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}]$.

Definition 7.2.5

Let $(\mathcal{M}, \mathcal{T}_1, \mathcal{T}_2)$ be a M-N-BTS and $\mathcal{A} \subseteq \mathcal{M}$ then the MN-Bi-boundary of $\mathcal{A}$ is denoted by $\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}$ and is defined as the intersection of the MN-Bi-closure of the mset $\mathcal{A}$ and the MN-Bi-closure of the complement of $\mathcal{A}$.

Thus, $\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd} = \mathcal{A}_{MN}^{\mathcal{T}_{12}-cl} \cap (c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}$

Proposition 7.2.6

For a M-N-B-TS $(\mathcal{X},\mathcal{T}_1,\mathcal{T}_2)$, *if* $\mathcal{A}\subseteq\mathcal{X}$, *then we have the following results:*

(i) $\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}=(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-bd}$

(ii) $\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int}\right)\cup((c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int})=c\left(\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}\right)$

Proof*:*

(i) We have: $(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-bd}=(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}\cap(c\{c\mathcal{A}\})_{MN}^{\mathcal{T}_{12}-cl}=(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}\cap\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}=\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\cap(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}=\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}$

(ii) We have: $c[\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}]=c[\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}\cap(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}]$

$=c[\mathcal{A}_{MN}^{\mathcal{T}_{12}-cl}]\cup c[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-cl}]$

$=[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}]\cup(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int})$,

[***proposition 7.2.5 (ii)*** and ***corollary 7.2.1 (i)***]

Thus, $c\left[\mathcal{A}_{MN}^{\mathcal{T}_{12}-bd}\right]=[(c\mathcal{A})_{MN}^{\mathcal{T}_{12}-int}]\cup(\mathcal{A}_{MN}^{\mathcal{T}_{12}-int})$

7.3 Multi-Anti-Topological Spaces

Definition 7.3.1

Let $\mathcal{M}\in[\mathcal{X}]^n$ *and* $\mathcal{T}\subseteq\mathcal{P}^*(\mathcal{M})$. *Then* $\mathcal{T}$ *will be called a multi-anti-topology (M-A-T) on* $\mathcal{M}$ *if it satisfies the following three properties:*

(i) *The m-set* $\mathcal{M}$ *and empty set* $\emptyset$ *are not in* $\mathcal{T}$.

(ii) *If* $\mathcal{A}_i\subseteq\mathcal{T}$, *then* $\cup\mathcal{A}_i\notin\mathcal{T}$

(iii) *If* $\mathcal{A}_i\subseteq\mathcal{T}$ *then* $\cap\mathcal{A}_i\notin\mathcal{T}$.

Then $(\mathcal{M},\mathcal{T})$ *will be called a Multi-Anti-Topological Space (M-A-TS) and the A-O msets of this space will be called Multi-Anti-Open msets or M-A-O msets and the m-complement of such msets will be called Multi-Anti-Closed msets or M-A-C msets.*

Example 7.3.1

If $\mathcal{X}=\{x,y,z,w\}$, $n=4$ and $\mathcal{M}=\{\frac{3}{x},\frac{2}{y},\frac{3}{z},\frac{4}{w}\}$.

Let $T=\{\emptyset,\{\frac{2}{x},\frac{1}{y}\},\{\frac{2}{x},\frac{1}{z}\},\{\frac{3}{y},\frac{1}{z}\},\{\frac{1}{z},\frac{2}{w}\},\ \{\frac{1}{x},\frac{2}{y},\frac{1}{z}\},\{\frac{2}{y},\frac{2}{z},\frac{1}{w}\},\{\frac{2}{z},\frac{2}{w}\}\}$.

Here $\emptyset\notin\mathcal{T}$ and $\mathcal{M}\notin\mathcal{T}$.And it can be checked for all submsets $\mathcal{A}_i\in\mathcal{T}$, $\cup\mathcal{A}_i\notin\mathcal{T}$ and for any submsets $\mathcal{A}_i\in\mathcal{T}$, $\cap\mathcal{A}_i\notin\mathcal{T}$.

Hence, $(\mathcal{M},\mathcal{T})$ is a *M-A-TS* on $\mathcal{M}$ over the set $\mathcal{X}$.

Proposition 7.3.1

Let $(\mathcal{M},\mathcal{T})$ *be a M-A-TS and* $\mathcal{A} \in \mathcal{T}$ *then* $\mathcal{B} \notin \mathcal{T}$ *if* $\mathcal{A} \subset \mathcal{B}$ *or* $\mathcal{B} \subset \mathcal{A}$*.*

Proof:

We have $\mathcal{A} \subset \mathcal{B}$ implies $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$ and $\mathcal{B} \in \mathcal{T}$ implies $\mathcal{A} \cup \mathcal{B} \in \mathcal{T}$ which is contradictory to the definition of a *M-A-T*. Hence $\mathcal{B} \notin \mathcal{T}$ whenever $\mathcal{A} \subset \mathcal{B}$.

Again, $\mathcal{B} \subset \mathcal{A}$ implies $\mathcal{A} \cup \mathcal{B} = \mathcal{A}$ and since $\mathcal{A} \in \mathcal{T}$ so $\mathcal{A} \cup \mathcal{B} \in \mathcal{T}$ which is also not possible for a *M-A-T*.

Definition 7.3.2

Let $(\mathcal{M},\mathcal{T})$ *be a M-A-TS formed on a mset* $\mathcal{M}$ *and* $\mathcal{A} \subseteq \mathcal{M}$*. Then the MA-Interior (MA-Int, in short) of* $\mathcal{A}$ *will be the mset union of M-A-O submsets of* $\mathcal{A}$ *and will be denoted by* $\mathcal{A}^{MA-Int}$.

Example 7.3.2

Let $\mathcal{X} = \{x, y, z, w\}, n = 4$ and $\mathcal{M} = \{4/x, 4/y, 3/z, 4/w\}$. We define a *M-A-T* on $\mathcal{M}$ as*:* $\mathcal{T} = \{\{\frac{1}{x}, \frac{2}{y}\}, \{\frac{2}{x}, \frac{1}{y}\}, \{\frac{1}{x}, \frac{1}{y}, \frac{2}{z}\}, \{\frac{2}{x}, \frac{1}{z}\}, \{\frac{2}{y}, \frac{1}{z}, \frac{2}{w}\}\}$. Suppose that $\mathcal{A} = \{\frac{2}{x}, \frac{2}{y}, \frac{2}{z}\}$ then $\mathcal{A}^{MA-Int} = \left\{\frac{1}{x}, \frac{2}{y}\right\} \cup \left\{\frac{2}{x}, \frac{1}{y}\right\} \cup \left\{\frac{1}{x}, \frac{1}{y}, \frac{2}{z}\right\} \cup \left\{\frac{2}{x}, \frac{1}{z}\right\} = \left\{\frac{2}{x}, \frac{2}{y}, \frac{2}{z}\right\} = \mathcal{A}$.

Remark 7.3.1

The *MA-Int* of a submset $\mathcal{A}$ is not the largest *M-A-O* submset contained in $\mathcal{A}$. This is because in a *M-A-TS* the union of *M-A-O* submsets is not *M-A-O*. In the above example it may be observed that the mset $\mathcal{A}$ is not *M-A-O*.

Proposition 7.3.2

For any submset $\mathcal{A}$ *of some M-A-TS* $(\mathcal{M},\mathcal{T})$, $\mathcal{A}^{MA-int} \subseteq A$. *Let* $\mathcal{A}$ *be M-A-O then* $\mathcal{A}^{MA-int} = \mathcal{A}$.

Remark 7.3.2

The converse of ***proposition 7.3.2*** is not always true. That is, if $\mathcal{A}^{MA-int} = \mathcal{A}$ then the mset $\mathcal{A}$ need not be *M-A-O*. This can be seen from ***example 7.3.2.***

Proposition7.3.3

If $\mathcal{A}$ *and* $\mathcal{B}$ *are submsets of some M-A-TS* $(\mathcal{M},\mathcal{T})$*, then we have the following*:

(i) $(\mathcal{A}^{MA-Int})^{MA-Int} = \mathcal{A}^{MA-Int}$

(ii) $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A}^{MA-Int} \subseteq \mathcal{B}^{MA-Int}$

(iii) $(\mathcal{A} \cap \mathcal{B})^{MA-Int} \subseteq \mathcal{A}^{MA-Int} \cap \mathcal{B}^{MA-Int}$

(iv) $\mathcal{A}^{MA-Int} \cup \mathcal{B}^{MA-Int} \subseteq (A \cup B)^{MA-Int}$

Proof:

(i) By definition, $\mathcal{A}^{MA-Int} = \cup \{O_i$: each O_i is *M-A-O* and $O_i \subseteq \mathcal{A}\} = \mathcal{A}$ (say)

Now, $(\mathcal{A}^{MA-Int})^{MA-Int} = \cup \{O_i$: each O_i is *M-A-O* and $O_i \subseteq \mathcal{A}^{MA-int}\} = \mathcal{B}$ since there will be no other *M-A-O* msets that will be extra from those subsets which are contained in $\mathcal{A}^{MA-int}$.

Thus, we must have: $(\mathcal{A}^{MA-Int})^{MA-Int} = \mathcal{A}^{MA-Int}$

(ii) Let $x \in \mathcal{A}^{MA-Int}$ then by ***proposition 7.3.2,*** $x \in \mathcal{A}$ and since $\mathcal{A} \subseteq \mathcal{B}$, so by ***proposition 7.3.2,*** $\mathcal{A} \subseteq \mathcal{B}^{MA-Int}$ and hence $x \in \mathcal{B}^{MA-Int}$.

So, $\mathcal{A}^{MA-Int} \subseteq \mathcal{B}^{MA-Int}$.

(iii) Since $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ as well as $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ together with *(ii)* above it can be deduced that $(\mathcal{A} \cap \mathcal{B})^{MA-Int} \subseteq \mathcal{A}^{MA-Int}$ and $(\mathcal{A} \cap \mathcal{B})^{MA-Int} \subseteq \mathcal{B}^{MA-Int}$ and hence the result.

(iv) Let $x \in \mathcal{A}^{MA-Int} \cup \mathcal{B}^{MA-Int}$.

Then $x \in \mathcal{A}^{MA-Int}$ or $x \in \mathcal{B}^{MA-Int}$

$\Rightarrow x \in \mathcal{U}, \mathcal{U} \subseteq \mathcal{A}$ or, $x \in \mathcal{V}, \mathcal{V} \subseteq \mathcal{B}$ and $\mathcal{U}$ and $\mathcal{V}$ are *M-A-O*.

$\Rightarrow x \in \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U} \cup \mathcal{V} \subseteq \mathcal{A} \cup \mathcal{B}$

$\Rightarrow x \in (\mathcal{U} \cup \mathcal{V})^{MA-Int} \subseteq (\mathcal{A} \cup \mathcal{B})^{MA-Int}$ by *(ii)*.

Hence, $x \in (\mathcal{A} \cup \mathcal{B})^{MA-Int}$.

Thus $\mathcal{A}^{MA-Int} \cup \mathcal{B}^{MA-Int} \subseteq (A \cup B)^{MA-Int}$

Remark 7.3.3

The converse will not necessarily hold in ***proposition 7.3.3 (iii)*** as shown by the following: Consider $\mathcal{X} = \{a, b, c, d\}$, $n = 3$ and $\mathcal{M} = \{\frac{3}{a}, \frac{3}{b}, \frac{2}{c}, \frac{3}{d}\}$. We define a *M-A-T* $\mathcal{T} = \{\{\frac{1}{a}, \frac{2}{b}\}, \{\frac{2}{a}, \frac{1}{b}\}, \{\frac{1}{a}, \frac{1}{b}, \frac{2}{c}\}, \{\frac{2}{a}, \frac{1}{c}\}, \{\frac{2}{b}, \frac{1}{c}, \frac{2}{d}\}\}$ on $\mathcal{M}$. Suppose that $\mathcal{A} = \{\frac{2}{a}, \frac{2}{b}, \frac{1}{c}\}$ and $\mathcal{B} = \{\frac{1}{a}, \frac{2}{b}, \frac{2}{c}, \frac{2}{d}\}$. Then $\mathcal{A} \cap \mathcal{B} = \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\}$ and $(\mathcal{A} \cap \mathcal{B})^{MA-Int} = \{\frac{1}{a}, \frac{2}{b}\}$ and $\mathcal{A}^{MA-Int} = \{\frac{2}{a}, \frac{2}{b}, \frac{1}{c}\}$ and $\mathcal{B}^{MA-Int} = \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}, \frac{2}{d}\}$ and thus we have $\mathcal{A}^{MA-Int} \cap \mathcal{B}^{MA-Int} = \{\frac{2}{a}, \frac{2}{b}, \frac{1}{c}\} \cap \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}, \frac{2}{d}\} = \{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\}$ and it can be seen that $\{\frac{1}{a}, \frac{2}{b}, \frac{1}{c}\} \notin \{\frac{1}{a}, \frac{2}{b}\}$.

Similarly, it can be shown that the converse of ***proposition 7.3.3 (iv)*** does not hold.

Definition 7.3.3

Let $(\mathcal{M},\mathcal{T})$ be a M-A-TS formed on a mset $\mathcal{M}$ and $\mathcal{B}$ be a submset of $\mathcal{M}$ then the MA-Exterior of $\mathcal{A}$ will be the MA-Interior of m-complement of $\mathcal{A}$ and will be denoted by $\mathcal{A}^{MA-Ext}$. That is $\mathcal{A}^{MA-Ext} = (c\mathcal{A})^{MA-Int}$ $=\cup \{O_i$: each O_i is M-A-O and $O_i \subseteq c\mathcal{A}\}$.

Proposition 7.3.4

If $\mathcal{A}$ and $\mathcal{B}$ are submsets of a M-A-TS $(\mathcal{M},\mathcal{T})$, then we have the following:

(i) $\mathcal{A}^{MA-Ext} = c\mathcal{A}$, *if* $\mathcal{A}$ *is M-A-C.*

(ii) $\mathcal{A}^{MA-Int} \cap \mathcal{A}^{MA-Ext} = \emptyset$

(iii) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathcal{A}^{MA-Ext} \supseteq \mathcal{B}^{MA-Ext}$

(iv) $(A \cup B)^{MA-Ext} \subseteq \mathcal{A}^{MA-Ext} \cap \mathcal{B}^{MA-Ext}$

(v) $\mathcal{A}^{MA-Ext} \cup \mathcal{B}^{MA-Ext} \subseteq (A \cap B)^{MA-Ext}$

Proof:

(i) If $\mathcal{A}$ is *M-A-C*, then $c\mathcal{A}$ is *M-A-O* and as such by ***proposition 7.3.2***: $\mathcal{A}^{MA-Ext} = (c\mathcal{A})^{MA-Int} = c\mathcal{A}$.

(ii) Let $x \in \mathcal{A}^{MA-Int} \cap \mathcal{A}^{MA-Ext} \Rightarrow x \in \mathcal{A}^{MA-Int}$ and $x \in \mathcal{A}^{MA-Ext}$

$\Rightarrow x \in \mathcal{A}$ and $x \in c\mathcal{A}$ which is not possible.

Hence $\mathcal{A}^{MA-Int} \cap \mathcal{A}^{MA-Ext} = \emptyset$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c\mathcal{B} \subseteq c\mathcal{A} \Rightarrow (c\mathcal{B})^{MA-Int} \subseteq (c\mathcal{A})^{MA-Int}$

[by ***proposition 7.3.3 (ii)***]

$\Rightarrow \mathcal{B}^{MA-Ext} \subseteq \mathcal{A}^{MA-Ext}$

(iv) $(\mathcal{A} \cup \mathcal{B})^{MA-Ext} = (c(\mathcal{A} \cup \mathcal{B}))^{MA-Int}$

$= (c\mathcal{A} \cap c\mathcal{B})^{MA-Int}$

$\subseteq (c\mathcal{A})^{MA-Int} \cap (c\mathcal{B})^{MA-Int}$ [by ***proposition 7.3.3 (iii)***]

$= \mathcal{A}^{MA-Ext} \cap \mathcal{B}^{MA-Ext}$

Hence, $(\mathcal{A} \cup \mathcal{B})^{MAE} \subseteq \mathcal{A}^{MAE} \cap \mathcal{B}^{MAE}$

(v) $\mathcal{A}^{MA-Ext} \cup \mathcal{B}^{MA-Ext} = (c\mathcal{A})^{MA-Int} \cup (c\mathcal{B})^{MA-Int}$

$\subseteq (c\mathcal{A} \cup c\mathcal{B})^{MAI}$ by ***proposition 7.3.3 (iv)***

$= (c(\mathcal{A} \cap \mathcal{B}))^{MA-Int}$

$= (\mathcal{A} \cap \mathcal{B})^{MA-Ext}$

Hence, $\mathcal{A}^{MA-Ext} \cup \mathcal{B}^{MA-Ext} \subseteq (A \cap B)^{MA-Ext}$

Definition 7.3.4

Let $(\mathcal{M},\mathcal{T})$ be a MATS formed on a set $\mathcal{M}$ and $\mathcal{A}$ be a submset of $\mathcal{M}$ then the MA-Closure of $\mathcal{A}$ will be the intersection of the M-A-C submsets that contain $\mathcal{A}$ and will be denoted by $\mathcal{A}^{MA-Cl}$. That is, $\mathcal{A}^{MA-Cl} = \cap\{\mathcal{C}_i$: each $\mathcal{C}_i$ is M-A-C and $\mathcal{A} \subseteq \mathcal{C}_i\}$.

Remark 7.3.4

MA-Closure of a submset $\mathcal{A}$ need not be the smallest *M-A-C* submset that contain $\mathcal{A}$. This is because in a *M-A-TS* the intersection of *M-A-C* submsets is not *MAC*.

Proposition 7.3.5

For any submset $\mathcal{A}$, $\mathcal{A} \subseteq \mathcal{A}^{MA-Cl}$. If $\mathcal{A}$ is a *M-A-C* submset then $\mathcal{A}^{MA-Cl} = \mathcal{A}$.

Remarks 7.3.5

The converse of ***proposition 7.3.5*** need not be true. That is, if $\mathcal{A}^{MA-Cl} = \mathcal{A}$ then $\mathcal{A}$ need not be *M-A-C*. The following example may be considered.

Let $\mathcal{X} = \{x, y, z, w\}, n = 4$ and $\mathcal{M} = \{4/x, 4/y, 3/z, 4/w\}$. We define a *M-A-T* on $\mathcal{M}$ as: $\mathcal{T} = \{\{\frac{1}{x}, \frac{2}{y}\}, \{\frac{2}{x}, \frac{1}{y}\}, \{\frac{1}{x}, \frac{1}{y}, \frac{2}{z}\}, \{\frac{2}{x}, \frac{1}{z}\}, \{\frac{2}{y}, \frac{1}{z}, \frac{2}{w}\}\}$. The *M-A-C* msets here are: $\left\{\frac{3}{x}, \frac{2}{y}, \frac{3}{z}, \frac{4}{w}\right\}, \left\{\frac{2}{x}, \frac{3}{y}, \frac{3}{w}, \frac{4}{z}\right\},, \left\{\frac{3}{x}, \frac{3}{y}, \frac{1}{z}, \frac{4}{w}\right\}, \left\{\frac{2}{x}, \frac{4}{y}, \frac{2}{z}, \frac{4}{w}\right\}, \{\frac{4}{x}, \frac{2}{y}, \frac{2}{z}, \frac{2}{w}\}$

Let $\mathcal{A} = \{\frac{2}{x}, \frac{2}{y}, \frac{1}{z}, \frac{2}{w}\}$ then $\mathcal{A}^{MA-Cl} = \{\frac{2}{x}, \frac{2}{y}, \frac{1}{z}, \frac{2}{w}\} = \mathcal{A}$. But $\mathcal{A}$ is itself not *M-A-C*.

Proposition 7.3.6

If $\mathcal{A}$ and $\mathcal{B}$ are submsets of a M-A-TS $(\mathcal{M},\mathcal{T})$ defined on a set $\mathcal{X}$, then the following results hold:

- *(i)* $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{MA-Cl} \subseteq \mathcal{B}^{MA-Cl}$.
- *(ii)* $(A \cap B)^{MA-Cl} \subseteq \mathcal{A}^{MA-Cl} \cap \mathcal{B}^{MA-Cl}$
- *(iii)* $\mathcal{A}^{MA-Cl} \cup \mathcal{B}^{MA-Cl} \subseteq (A \cup B)^{MA-Cl}$.
- *(iv)* $(\mathcal{A}^{MA-Cl})^{MA-Cl} = \mathcal{A}^{MA-Cl}$.

Proof:

(i) By ***proposition 7.3.5***, we have: $\mathcal{A} \subseteq \mathcal{A}^{MA-Cl}$.

Now, $\mathcal{A}^{MA-Cl} = \cap\{\mathcal{C}_i$: each $\mathcal{C}_i$ is *M-A-C* and $\mathcal{A} \subseteq \mathcal{C}_i\}$

$\subseteq \cap\{\mathcal{D}_i$: each $\mathcal{D}_i$ is *M-A-C* and $\mathcal{B} \subseteq \mathcal{D}_i\}$ since $\mathcal{A} \subseteq \mathcal{B}$, each $\mathcal{C}_i \subseteq \mathcal{D}_i$

$= \mathcal{B}^{MA-Cl}$, and hence the result.

(ii) Since $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{MA-Cl} \subseteq \mathcal{A}^{MA-Cl}$, by *(i)*.

Also, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{MA-Cl} \subseteq \mathcal{B}^{MA-Cl}$, by *(i)*.

Hence, $(\mathcal{A} \cap \mathcal{B})^{MA-Cl} \subseteq \mathcal{A}^{MA-Cl} \cap \mathcal{B}^{MA-Cl}$.

(iii) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{A}^{MA-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MA-Cl}$.

Also, $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{B}^{MA-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MA-Cl}$.

Hence, $\mathcal{A}^{MA-Cl} \cup \mathcal{B}^{MA-Cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{MA-Cl}$.

(iv) Let $\mathcal{A}^{MA-Cl} = \mathcal{B}$, then $\mathcal{B}^{MA-Cl} = \cap \{\mathcal{C}_i$ so that each $\mathcal{C}_i$ is *M-A-C* and $\mathcal{B} \subseteq \mathcal{C}_i\} = \mathcal{B}$, since $\mathcal{B}$ is the smallest submset containing $\mathcal{A}$ and there can be no smaller submset of $\mathcal{B}$ that contains $\mathcal{A}$ which can be contained in the intersection $\cap\, \mathcal{C}_i$ and so: $(\mathcal{A}^{MA-Cl})^{MA-Cl} = \mathcal{A}^{MA-Cl}$.

Proposition 7.3.7

For any submset $\mathcal{A}$ of a M-A-TS $(\mathcal{M}, \mathcal{T})$, the following are true:

(i) $\mathcal{A}^{MA-Int} = c((c\mathcal{A})^{MA-Cl})$

(ii) $(c\mathcal{A})^{MA-Cl} = c(\mathcal{A}^{MA-Int})$

(iii) $c(\mathcal{A}^{MA-Cl}) = (c\mathcal{A})^{MA-Int}$

(iv) $\mathcal{A}^{MA-Cl} = c((c\mathcal{A})^{MA-Int})$

Proof:

(i) We have: $(c\mathcal{A})^{MA-Cl} = \cap\, \mathcal{C}_i$, where each $\mathcal{C}_i$ is *M-A-C* and $c\mathcal{A} \subseteq \mathcal{C}_i$.

Thus, $c((c\mathcal{A})^{MA-Cl}) = c(\cap\, \mathcal{C}_i)$ so that $c(c\mathcal{A}) \supseteq c\mathcal{C}_i$

Or, $c((c\mathcal{A})^{MA-Cl}) = \cup\, c\mathcal{C}_i$ so that each $c\mathcal{C}_i$ is *M-A-O* and $c\mathcal{C}_i \subseteq \mathcal{A}$

Or, $c((c\mathcal{A})^{MA-Cl}) = \cup\, \mathcal{O}_i$ so that each $\mathcal{O}_i$ is *M-A-O* and $\mathcal{O}_i \subseteq \mathcal{A}$

Or, $c((c\mathcal{A})^{MA-Cl}) = \mathcal{A}^{MA-Int}$

(ii) We have: $\mathcal{A}^{MA-Int} = \cup\, \mathcal{O}_i$, where each $\mathcal{O}_i$ is *M-A-O* and $\mathcal{O}_i \subseteq \mathcal{A}$

So, $c(\mathcal{A}^{MA-Int}) = c(\cup\, \mathcal{O}_i)$ so that $c\mathcal{O}_i \supseteq c\mathcal{A}$

Or, $c(\mathcal{A}^{MA-Int}) = \cap\, c\mathcal{O}_i$ so that each $c\mathcal{O}_i$ is *M-A-C* and $c\mathcal{A} \subseteq c\mathcal{O}_i$

Or, $c(\mathcal{A}^{MA-Int}) = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *M-A-C* and $c\mathcal{A} \subseteq \mathcal{C}_i$

Or, $c(\mathcal{A}^{MA-Int}) = (c\mathcal{A})^{MA-Cl}$.

(iii) We have: $\mathcal{A}^{MA-Cl} = \cap\, \mathcal{C}_i$, where each $\mathcal{C}_i$ is *M-A-C* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Hence $c(\mathcal{A}^{MA-Cl}) = c(\cap\, \mathcal{C}_i)$, where each $c\mathcal{C}_i$ is *M-A-O* and $c\mathcal{A} \supseteq c\mathcal{C}_i$

Or, $c(\mathcal{A}^{MA-Cl}) = \cup\, c\mathcal{C}_i$ where each $c\mathcal{C}_i$ is *M-A-O* and $c\mathcal{C}_i \subseteq c\mathcal{A}$.

Or, $c(\mathcal{A}^{MA-Cl}) = \cup\, \mathcal{O}_i$ where each $\mathcal{O}_i$ is *M-A-O* and $\mathcal{O}_i \subseteq c\mathcal{A}$.

Or, $c(\mathcal{A}^{MA-Cl}) = c(\mathcal{A}^{MA-Int})$

(iv) We have: $(c\mathcal{A})^{MA-Int} = \cup\, \mathcal{O}_i$, where each $\mathcal{O}_i$ is *M-A-$\mathcal{O}$* and $\mathcal{O}_i \subseteq c\mathcal{A}$

Thus, $c((c\mathcal{A})^{MA-Int}) = c(\cup\, \mathcal{O}_i)$ so that $c\mathcal{O}_i \supseteq c(c\mathcal{A})$

Or, $c((c\mathcal{A})^{MA-Int}) = \cap\, c\mathcal{O}_i$ so that each $c\mathcal{O}_i$ is *M-A-$\mathcal{C}$* and $\mathcal{A} \subseteq c\mathcal{O}_i$

Or, $c((c\mathcal{A})^{MA-Int}) = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *M-A-$\mathcal{C}$* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Or, $c((c\mathcal{A})^{MA-Int}) = \mathcal{A}^{MA-Cl}$.

Definition 7.3.5

Consider $(\mathcal{M}, \mathcal{T})$ to be a M-A-TS defined on a mset $\mathcal{M}$, where $\mathcal{M}$ is defined over an arbitrary universe $\mathcal{X}$ and $\mathcal{A}$ is a submset of $\mathcal{M}$. Then the MA-Boundary of $\mathcal{A}$ is defined as the intersection of the MA-Closure of $\mathcal{A}$ and the MA-Closure of the m-complement of $\mathcal{A}$ and will be denoted by $\mathcal{A}^{MA-Bd}$. That is, $\mathcal{A}^{MA-Bd} = \mathcal{A}^{MA-Cl} \cap (c\mathcal{A})^{MA-Cl}$.

Proposition 7.3.8

If $\mathcal{A}$ is a submset of a M-A-TS $(\mathcal{M}, \mathcal{T})$ where the mset $\mathcal{M}$ is defined over a set $\mathcal{X}$, then $c(\mathcal{A}^{MA-Bd}) = \mathcal{A}^{MA-Int} \cup \mathcal{A}^{MA-Ext}$.

Proof:

We have: $c(\mathcal{A}^{MA-Bd}) = c(\mathcal{A}^{MA-Cl} \cap (c\mathcal{A})^{MA-Cl}) = c(\mathcal{A}^{MA-Cl}) \cup c((c\mathcal{A})^{MA-Cl})$

$= (c\mathcal{A})^{MA-Int} \cup \mathcal{A}^{MA-Int}$ [by ***proposition 7.3.7 (iii) and (i)***]

$= \mathcal{A}^{MA-Int} \cup \mathcal{A}^{MA-Ext}$