CHAPTER 1

Introduction, Literature Review, and Basic Concepts

This chapter deals with the evolution of topological spaces from the initial categorization and foundation of set-theoretic topology till the present day where various spaces are defined and studied on the basis of the new types of sets that have been defined over the period and also deals with the related aspects that will be undertaken in the research work.

1.1 Introduction

Point set topology, alsoreferred to as general topology, can be called the study of set theoretic ideas and how to manipulate or construct them for application in topology or the study of different set structures.A framework for examining and comprehending the characteristics of spaces that are independent of particular geometric features like distance and angles is provided by general topology. One type of instrument or tool employed in the study and analysis of metric spaces, which can possibly be considered as the forerunner to the study of topological spaces, is distance. According to common understanding, topology is the area of mathematics that studies the characteristics of a geometric object that hold true while being repeatedly stretched, twisted, crumpled and bent without snapping, breaking or creating holes in it. Because the structures or objects under study in topology can be stretched, compressed, and deformed like rubber but cannot be torn apart or pierced to create holes in them, the subject is sometimes referred to as rubber-sheet geometry. When objects are distorted from one form to another through allowable manipulations, homeomorphism between the objects is considered to be the usual method for preserving their topological attributes or properties. Homeomorphism, coming from Greek, which means having identical form or shape, also known as structural isomorphism, is characterized by continuous function between spaces that also has continuous inverse function. With a long history, the field of general topology has been substantially evolved over the years. Even though there aren't many similarities across the various disciplines of topology, general topology serves as the basis for majority of them, including differential topology, geometric topology, and algebraic topology. Many fields of research have given rise to general topology; the majority of them are related to the in-depth examination of the subsets of the real line and their characteristics. The evolution can also be attributed to the study of metric spaces and normed linear spaces, as well as the advent of the idea of manifolds. Although the origins of the study of topology cannot be directly attributed to any one person, Leonhard Euler and his Seven Bridges of Königsberg problem are frequently cited as the starting point. Euler introduced the concept of the Euler characteristic for polyhedral in the 18th century, which paved the way for the development of combinatorial topology, which was centered on using combinatorial methods to understand the properties of geometric objects. Combinatorial topology is basically the topological study of geometric figures or objects by considering them as elementary geometric figures such as straight lines or individual points. The formula for a polyhedron and Euler's Königsberg seven bridges problem are acknowledged to be the earliest results in the area of topology. From the formula of a polyhedral, the results were later applied to other objects with holes in them to obtain distinct and unique formulae for objects that were comparable. The concept of connectedness of points took the center of study rather than the distance between the points in the analysis of the problem. Henri Poincare, who developed the concepts of homology and homotopy, which are essential tools in the discipline of topology, made substantial contributions to the field in the late 1800's and early 1900's by adding algebraic invariants to the study of topological spaces. A fundamental feature of algebraic topology is homology, which allows one to differentiate between a surface's inner and outer sides even when both surfaces may share come features but still have other unique qualities. In algebraic topology, homotopy is commonly defined as the situation when two curves with common end points are drawn on an object's surface and one of the curves can continuously be deformed into the other, leaving the endpoints. The formalization of the idea of topological spaces in the 20th century gave rise to point-set topology, which is specialized in the study of open sets, closed sets, and continuous functions. This framework allowed for a more comprehensive approach studying the conceptions of continuity, convergence, and compactness in topology. Geometric topology and differential topology are further subfields of the field that have grown throughout time.

The latter is concerned with the study of geometric structures on topological spaces, while the former works with smooth manifolds and differential mappings. These branches of topology have stronglinks to differential geometry and have led to many important results, one of which is the Poincare conjecture which is considered to be the first of its kind in the field and could be solved about a hundred years later in 2003 from the year 1904 when it was conjectured. The study of topological data analysis, which uses topological techniques to analyze large, complicated data sets, has grown in popularity in recent years. The multidisciplinary discipline has applications in material science, machine learning, and neuroscience, among other fields. All things considered, the growth of topology has been characterized by the creation of fresh ideas, methods, and applications that have improved our comprehension of a space's structure and given researchers access to effective resources for handling issues in mathematics and other academic disciplines. The study of topology started with the study of dimensions. The dimensions related to a point, a line and a cuboid which were accepted to be the study in one dimension, two dimension and three dimensions respectively. Attempts were also made to establish studies in higher dimensions beyond three, which contributed to the origin of manifolds in topology. The Cantorian theory of sets added wings to the study of topology leading to the evolution of the set-theoretic topology or the general topology of the present day. The axiomatic definition of set-theoretic topology which is based on open sets was initiated by a class of French mathematicians, known as Nicolas Bourbaki collectively or pseudonym as the Bourbaki Group. The definition was formulated during the years 1935 to 1938 by the Bourbaki Group, by amalgamatingand deeply analyzing the studies made by the mathematicians Giuseppe Peano (1858-1932), Felix Hausdorff (1868-1942), Hermann Weyl (1885-1955), Frigyes Riesz (1880-1956), René Maurice Fréchet (1878-1973), and Pavel Sergeyevich Aleksandrov (1896-1982). The openness of the random union and limited intersection of subsets served as the foundation for the original notion of a topology. Open sets are basically subsets of a set under study. However, later the openness of the void set and the set itself, were added to the axioms in 1939 by the Bourbaki and hence we have the present-day definition of topology or a topological space. General topology or topology has been defined on a set to be a group of the subsets of the collection of all possible subsets, or the group of all subsets, of the set in such a way that the void set and the set itself belongs to the collection, arbitrary unions of the subsets belong to the collection and finite intersections of the subsets belong to the collection. The set that is considered in the definition of general topological space (GTS) is the classical set, where all the members of the set are well defined. However, over the years, it has been found that the classical set is not enough to define the notion of many a collection of objects. At times, the very concept of a collection of objects failed to cover all the objects that deemed to be a member of the named collection. Numerous such collections of objects, items, people etc. defy the notion of the classical set and as such defining a topological space over such vague collection of objects made the notion devoid of the actual concept. The classical set fails to cover aspects that are not properly defined. But, in many situations, unclear collections occur in real-life situations, without clear boundaries or limits or even vivid starting points. As such, many scholars and philosophers contemplated on refining the very idea of a set, which would possibly encompass all the possible members of all envisaged collection of objects, items, people and so on. Such searches for newer ideas brought about the concepts of a fuzzy set, which helped in taking care of ill-defined sets, the intuitionistic fuzzy set, by which the scope of the fuzzy set was improved, the neutrosophic set, which further generalized the intuitionistic fuzzy set, and many other refinements of the sets have been developed over time to meet the conditions of the collection of objects failed by the classical set. The Cantorian set is the basis to the study of point set or general topology or topology, and as such corresponding to the definition of the new types of sets, new definitions of topological spaces specific to the new sets evolved. An element in the classical set is classified according to whether it belongs to the set or not. Put otherwise, an element's membership in the classical set can be either 1 or 0, based on whether it is inside the set or outside the set. However, elements of a fuzzy set have assorted membership grades, ranging from 0 to 1, inclusive of all points in the range. Fuzzy topological space was defined in accordance with the fuzzy set, and researchers and scholars in the field have conducted several studies in the area. The basis of the fuzzy set is a membership of some degree which automatically suggests non-membership for all members having degree of membership less than 1 or perfect membership. However, the non-membership grade was missing in the early definition of a fuzzy set and was later added to the fuzzy set as a component and renamed as intuitionistic fuzzy set and subsequently its topology and space was defined. After the advent of fuzzy set, and subsequent development and integration of intuitionistic fuzzy set, besides their corresponding topological spaces, came the neutrosophic set, where the set was partitioned into three distinct components: the truth, the indeterminate and the falsity. Definition of the neutrosophic fuzzy set induced the definition of neutrosophic topology and its space. As a consequence of the neutrosophic set new structures called neutro-structures and anti-structures have been defined and studied. The studies made on the neutro and anti-structures brought along neutro and anti-geometry, neutro and anti-algebra and many other concepts. As a consequence, to the study of the neutro and anti-structures, the notion of neutro-topology and anti-topology was developed by academicians in 2021. The definitions of the neutro-topology and the anti-topology are defined on the classical set. In this thesis, some properties of GTSs are studied with the neutro-topology and the anti-topology. It has been found that many properties of the interior, the exterior, the closure and the boundary that are true in GTSs are also true in the two new topologies under study, while a few are found to be valid only with certain additional conditions on the open sets considered in the two new topologies. However, in the case of certain properties, no links could be established even with additional conditions that seemed exists in those particular cases. However, further studies on the same could be done outside the limit of this study to establish and find out such conditions, for which those results also could be established. It took generations of mathematicians to reach a consensus on even the current definition of a topological space, allowing the study that had been ongoing for generations to take the form of a single agreeable shape. What has finally been understood in the current study is that loosening classical ideas need deeper analysis for each classically established result. More rigorous analysis might be required to finally establish all the missing links in some of the results of this study.

1.2 Literature Review

General topology is a vast field of study that have been established by many mathematicians over the past few generations and over the previous century, a lot of study has been added to the study of the subject besides adding a lot of extensions, and generalizations to the initial concepts of the subject in the initial establishment. Cantor, who first described the concept of a derived set in 1872, can be credited for introducing the idea of limit points, closed and open intervals or subsets of real line. These concepts can be considered the cornerstones of modern set-theoretic topology. However, even in the current era of the internet, where abundant resources are available online in many different digitalized forms, it is very difficult to tract who first defined the interior, exterior, adherent point, limit point, closure, boundary etc. of a constituent member set of a topological space. It is very difficult to credit anyone with the initiation of these concepts in topological spaces. However, Giuseppe Peano (1858-1932), an Italian mathematician, can be credited for defining the terms interior point, exterior point, boundary in the analysis of an interval in the real line before the advent of the conception of open sets while defining a topology. The notion of interior, exterior, closure and boundary are all well laid concepts in point set topology and can be traced in all general topology books published by different authors over the years. The definitions of interior, closure and boundary are found in the textbook on General Topology by J. L. Kelley [**86**] first published in 1955. In the definition for a topological space provided by Kelley [**86**], he defined a topology on a set to satisfy two conditions which are the axioms of the union of subsets and the intersection of subsets. He added that the whole set and empty set naturally belongs to the topology as they both belong to the class of subsets of the universal set. The same type of approach of defining a topological space on the basis of only the two axioms of union and intersection of subsets can also be seen in [**59, 64, 153**]. However, in newer editions of [**64**] and from [**59**] the axiom of the inclusion of the void set and the whole set in the definition of a topology or topological space could be seen. In the book by Kelley [**86**], it has also been observed that the empty set was not denoted by the present-day commonly used notation or symbol "phi", instead all throughout the book, the empty or null set was denoted by the symbol "0". In an earlier article by Bing [**36**], on analyzing conditions for the metrization of topological spaces, the null set was denoted by the symbol "0", see page 185. Today, the empty set and the symbol "phi" or ∅are used synonymously in literature. However, in the article "Topologies on spaces of subsets", by Ernest Michael, published in "Transactions of the American Mathematical Society", Volume 71, Issue 1, pages: 152-182, in the year 1951, the empty set was denoted by the symbol "phi", page 153. Kelley's book was first published in the year 1955. However, it can be found in literature that the Bourbaki used the symbol ∅, which was a Danish symbol, to denote the empty set in 1939, which is a O with a slash, which eventually evolved to the present-day “phi”. The only reason “0” might have been used is explained from the fact that the character set of the English alphabet did not have the provision to write the symbol used by Bourbaki, when they first formalized the concept of point set topological space between 1935 and 1939. This fact is mentioned here only to infer that many things have evolved in the field of study of topological spaces over the years.

The concept of a bitopological space was brought about by Kelly [**87**] in 1963 who defined a bitopological space to be a space having two topologies induced by two asymmetric quasi-metrics. Patty [**127**], in 1967, furthered the study of bitopological spaces and confirmed some of the results stated by Kelly and provided sufficient counter-examples to prove that all metrization theorems that are true for a topological space cannot be true for a bitopological space because the two topologies that are considered to form the bitopological space do not form metric spaces simultaneously. Pervin [**128**] introduced the notion of connectedness in bitopological spaces along with the definition of continuity of functions between the spaces and analyzed six different results in connectedness. Notion of continuity between the spaces have been used to establish some of the results. Garner [**65**] defined bi-open sets in bitopological spaces and further defined bi-closed sets, bi-interior, bi-closure, and bi-separation axioms. The author used the concept of bi-open sets to study the hereditary properties of the bi-separation axioms and further investigated bi-continuity and bi-convergence with the use of the bi-open sets. Datta [**52**] introduced quasi-open and quasi-closed sets in bitopological spaces and studied quasi-closure and quasi-continuity to study semi-compactness properties in bitopological spaces. Further, studies of local connectedness, local compactness, pair-wise compactness, and separation axioms have been studied by many mathematicians and scholars in bitopological spaces [**48, 53, 133**].

Continuity of functions between two topological spaces is a well-established concept and has been defined in many ways. Kelley [**86**] characterized continuity of a function fromone topological space to another by eight equivalent definitions and characterizations. Validation of any one of the eight characterizations given by him is equivalent for a function betweentwo topological spacesbeing continuous. One such characterization is the closedness of the pre-image of closed sets which can be equivalently characterized with regard to the openness of the inverse image of open sets.

The fact that a topological space has been defined with regard to open sets and as such when only openness of sets is considered, a lot of studies has been done by various scholars in terms of different open sets characterized by different topological conditions and consequently the continuity of functions in terms of such open sets as defined by various scholars over the years. Halfar [**74**] in 1960 studied conditions that imply continuity of functions between spaces and studied continuity in terms of connectedness and compactness of the spaces. Levine [**95**–**96**] introduced the conception of weakly continuous functions by using the openness of the pre-image of open sets in the operators on closures of the two topologies he considered and studied some properties of weakly continuous functions and in the later article he introduced sets which were called semi-open and defined semi-continuity of functions in terms of the semi-open sets. Other studies that have been carried on semi-continuity can be seen in [**116, 117, 136**]. Studies on weakly, sub-weakly and semi-weakly continuous functions can be seen in [**25, 57, 62, 118, 123, 137**]. Hussain [**77**] in 1966 defined almost continuous functions in terms of openness of image of inverse image of the neighbourhood of a point. According to this concept, continuity of a function always implied almost continuity but not the other way around. Almost continuity has been studied by many other scholars [**98-100,137, 155**]. Gentry *et al.* [**67**] in 1971 introduced somewhat open sets and defined somewhat continuous functions and studied the properties of such continuous functions. Noiri [**119**] in 1980 introduced δ-closed sets and consequently its complement, the δ-open sets and subsequently defined the δ-continuous functions and proved that δ-continuity implies to weakly continuity. Mashour *et al*. [**105**] in 1982 defined pre-continuous and weakly pre-continuous functions on the basis of pre-open sets. And on the basis of the sets defined and named α-open sets, studies on the α-continuous functions have been done in [**107, 121, 122, 134, 175, 176**]. Many other scholars defined many different types of open sets and defined continuity with regard to those open sets but the list is inexhaustible as new types of open sets and new types of closed sets have been defined over the years and compiling all such open sets and the corresponding continuous functions will be cumbersome.

Pavel Urysohn (1898 – 1924) is credited for making the first ever systematic study on the axioms of separation in topological spaces. Aull *et al.* [**21**] in 1962 studied separation axioms between the T_0 space and the T_1 space and established certain conditions satisfying which, a topological space will be a T_0 topological space and also some other conditions satisfying which a topological space will be a T_1 topological space. They also introduced some three other separation axioms, which are of the weaker types, between T_0 and T_1. Further, Aull [**22**] in 1968 defined E_0 and E_1 spaces on the basis of the intersection of closed neighbourhoods and concluded that perfectly normal T_1 and T_2 spaces are necessarily E_1 spaces. Singhal [**154**] in 1971 analyzed the various separation axioms defined in topological spaces and attempted to depict the inter-relations between them. The study also covered separation axioms in bi-topological spaces and co-topological spaces. Maheswari *et al*. [**104**] in 1975 introduced axioms of separation in terms of the semi-open sets and named the separation axioms as s-separation axioms. Dorsett [**56**] in 1981 studied semi-separation axioms and concluded that the properties of semi-regularity, semi-normality are not semi-topological properties and that semi-regularity and semi-normality are independent axioms. Dube [**58**] in 1982 studied the properties of compactness, local compactness, almost compactness, and some inter-relations between those properties, in R_1 - topological spaces and establish some results on the hereditary properties of locally compact spaces. Caldas *et al.* [**43**] in 2003 conducted a study on low axioms of separation using the concept of the δ-open sets and δ-closure. Study on separation axioms in bitopological spaces have been done by various scholars some of which can be seen in [**19, 42, 108, 124, 131**].

The origins of the theory of multisets can be traced to many different sources and in different times, different terms like list, bunch, bag, heap etc. had been used by different scholars to mean the repetition of objects in a set, which in the present day is known as a multiset. For instance, in the article "On the theory of bags", by Yager [**187**] in 1986 where the term bag was used to represent some set-like objects, the elements in the bags had repetitions. In the same article he defined some basic operations of addition, union and intersection in the bag. Blizzard [**37**] in 1989 made a detailed study on multisets laying the foundation for the study on multisets, the type of set where repetition of the members is allowed, and replaced the other terms like bunch, bag, heap etc. for good. Blizzard used the shortened form mset to denote a multiset, defined multiplicity and cardinality of an mset and founded the binary axioms and algebraic operations on multisets enabling other scholars to expand the study in the domain of multisets. According to him a classical set is a multiset where multiplicity of each member is 1. Blizard [**38**] extended the study of multisets to real valued functions by restricting the multiplicity of the elements to the interval (0,1]. Blizard [**39**] further extendedthe study of multisets to the field of integers, where he proposed that the multiplicity in a multiset can be any integral value and may thus have negative multiplicity or negative membership in the multiset. He defined addition, subtraction, union, intersection and other algebraic axioms in multisets with integral multiplicities of elements. Petrovsky [**129**] in 1994 studied multisets and tried to define some metrics using a theoretical model with the help of multi-criteria decision-making techniques. Singh *et al*. [**157**], in 2007, mentioned some applications of multisets. Girish *et al*. [**69**] in 2009 formalized the definition of a multiset by suggesting certain notations and symbols and further introduced the notion of functions and relations in the study of multisets and established many existing results of functions and relations in calculus to the multiset setup. Singh *et al.* [**158**] in 2011 introduced some new operations on multisets and further delved into the difficulties in the operations of difference and complementation of multisets and also studied some applications of multisets. Ibrahim *et al.* [**79**] in 2011 introduced the notion of multiset space and operations on the multiset space algebra by introducing the operations of modulo in union, intersection and product of multisets. They further validated the algebraic operations of closure, commutativity, associativity, identity, existence of inverse element etc. in multisets. Girish *et al*. [**70**] in 2012 introduced multiset topological space and defined multiset basis for the multiset topology and further defined interior, closure, and their operators in the multiset topological spaces. They also defined neighbourhood and neighbourhood operator wherein they have considered the multisets as points. Girish *et al.* [**71**] in 2012 introduced the basis for a multiset topological space and named it as m-base and further introduced the concept of closed multisets and extended the study to establishing many results of interior, closure and limit points in multiset topological spaces, a few of which were disproved by Zakaria in [**193**]. However, they further introduced continuous multiset functions and established some results on the multiset functions. Babitha *et al.* [**24**] studied soft multi-sets. Zakaria [**193**] in 2015 provided a counter example to show otherwise one theorem and consequently one corollary that Girish *et al.* [**71**] had proved related to the closure and set of limit points in multiset topological spaces. El-Sheikh *et* *al.* [**60**] in 2015 studied separation axioms in multiset topological spaces by defining multiset separation axioms and compared results that are valid for separation axioms in GTSs and also studied the preservation of hereditary properties of the separation axioms they have defined. Das *et al.* [**49**] in 2016 added the concepts of exterior and boundary to the study of multiset topological space and established some results related to the concepts thereby enlarging the study on multiset topological spaces. Some work on multiset bitopological spaces can be seen in [**61, 150**]. Shravan *et al.* [**151**] in 2018 studied generalized closed sets in multiset topological spaces and called them generalized closed multisets and studied some of the properties that are valid in generalized closed sets in general topology. Kumar [**91**] in 2020 studied two kinds of connectedness with regard to multiset topological spaces and analyzed many results in the study. Shravan *et al.* **[152**] defined a metric between two multi-points in a finite multiset and studied the conditions for metrization of multiset topological spaces and laid down various results.

Deviating from the classical set, the first definition of the set to study fuzziness called a fuzzy set had been provided by Lofti Aliasker Zadeh [**190**] in 1965. In the classical set, the members of the set are well defined, and one can either be member or non-member of the set. But when identifying whether a particular entity is either a member or not of a set becomes unclear, then one cannot conclude that the entity belongs to the set or not in the strict sense. One can imagine a set, which has to be the collection of intelligent students. Such sets are not well defined because the word intelligent itself is not well defined. Such a set becomes a fuzzy set because intelligence will be measured by some criteria which will rank the students in order of the score they obtain in the testing criteria. Thus, in a fuzzy set the members possess a ranging membership degree defined between 0 and 1. The member of the fuzzy set with membership 0 will not be considered intelligent and the member with the membership 1 will be considered as the most intelligent, and others will have membership between 0 and 1. However, the lower and upper boundaries are not well foundedin a fuzzy set because the criteria which were used, if changed, may result in a different set. In a fuzzy set the logic is not only of true or false as in the case of the classical set where 0 would mean absolute non-membership and 1 would mean absolute membership. Fuzzy sets differ from classical probability theory in the sense that probability is an accurate measure between 0 and 1 but in context to a fuzzy set the membership values are vague and not specific. In probability theory, uncertainty is measured as a subset of a given collection of alternatives but in a fuzzy set membership in the set is not a measure but a degree of not being completely true or completely false. In his own words, Zadeh [**190**] defined the fuzzy set to be a: "class of objects with a continuum of grades of membership" [**190**]. He defined a fuzzy set to be characterized by an inclusion map that would assign every member a membership grade values from 0 to 1. He extended the operations of containment, union, intersection, complementation, relation, convexity etc. in the fuzzy set. He also defined the algebraic operation like sum, product and absolute difference between two fuzzy sets besides providing detailed interpretations to the operation he introduced to fuzzy sets. Goguen [**72**] in 1967 extended the study on the fuzzy set by adding the properties of a partial order set and called the fuzzy set as L-fuzzy set, where he replaced the unit interval of 0 to 1, by a partially ordered set or more specifically a lattice in [0, 1] and made an in-depth study on the fuzzy operations, relations, functions and founded many results. Chang [**44**] in the year 1968 with regard to the definition of fuzzy set and other concepts provided by Zadeh, introduced the fuzzy topology and fuzzy topological space (FTS) thereby introducing fuzzy-open and closed sets, neighbourhood of fuzzy point, interior, compactness and continuity in the fuzzy topology in line with properties that are valid for classical sets in GTSs. In general topology, neighbourhood is defined for a point but Chang [**44**] defined the fuzzy neighbourhood in a fuzzy set because a fuzzy point was not defined by Zadeh. Chang [**44**] defined continuity of functions in context of the openness of inverse image of open fuzzy sets in the fuzzy topology. Brown [**40**] in 1971 defined the membership function into the Boolean Lattice in [0, 1] instead of the unit interval itself, and studied the set theoretic properties proposed by Zadeh for the fuzzy sets and concluded that the set theoretic results hold true in the Boolean lattice also. Zadeh [**191**] in 1972 suggested some basic ideas to convert linguistic hedges into operators which can be acted on a fuzzy set and in the same article he discussed operations like fuzzification, intensification, accentuation etc. and introduced the fuzzy singleton set. De Luca *et al.* [**54**] in 1972 studied the nature of fuzzy set and that of L-fuzzy set and tried to analyze the algebraic properties of both the types of fuzzy sets in view of both the definitions and realized that generalizations of the algebraic properties were not possible till that time because of the inability to establish the complementation of the fuzzy set in the algebraic structure proposed by Zadeh for the fuzzy set. Nazaroff [**115**] in 1973 defined interior, exterior, closure and boundary in FTSs and discussed some results. Wong [**183**] in 1973 studied the different types of compactness in FTSs and found some variations in the results from those in GTSs. Wong [**184**] in 1974 introduced the concept of fuzzy point using the concept of fuzzy singleton sets initiated by Zadeh [**191**] and used it to study local countability, separability and local compactness and observed slight variations from established results in general topology because of which, results which looked simple required elaborate proofs. Convergence in FTS became more meaningful because of the definition of the fuzzy points. Wong [**185**] in 1975 extended the concepts of point set topology to the fuzzy topology and argued that the fuzzy set introduced by Zadeh followed similar operation like classical sets. Weiss [**179**] in 1975 studied FTSs induced by a fuzzy set and the subsets of the set which may be considered as naturally open for the fuzzy topology. He further studied the various properties of the fuzzy set that induced the fuzzy topology, and because of the induced topology, the properties that are true for the inducing fuzzy sets are also analyzed. Zadeh [**192**] in 1975 introduced the concept of linguistic variable in analyzing the fuzzy set theory and redefined another form of fuzzy set, which he named fuzzy set of type 2, where the grades of members in the membership function mapped to an ordinary fuzzy set in the unit interval 0 to 1. He also defined the various algebraic operations on the new type of fuzzy set and also various relations between the new type of fuzzy set. Mizumota *et al.* [**111**] in 1976 studied with detailed analysis, the characteristics of the algebraic operations with regard to the union, intersection, and complementation for the fuzzy set of type-2. They also tried to establish a partial order relation in the fuzzy set of type-2. Lake [**92**] in 1976 studied von Nuemann's axiomatization on the fuzzy sets and multisets. Warren [**177**] in 1977 studied the frontier of a fuzzy set and established many results for the frontier of a fuzzy set. Warren [**178**] in 1978 introduced the notion of neighbourhood of a point and using already founded concepts of sub-basis and closure, established six characterizations of functions that are continuous in FTSs and further stated that the fuzzy topology is a natural generalization of set theoretic topology. Conrad [**47**] in 1980 studied continuity in the fuzzy topology and introduced fuzzy filters and characterized convergence in the fuzzy topology with the help of fuzzy filters and further established equivalence between convergence and continuity. Ming *et al.* [**126**] in 1980 tried to redefine a fuzzy point to take the form of a singleton crisp set or anyusual point and introduced a connection between the fuzzy point and the fuzzy sets which they named the Q-relation and defined a corresponding neighborhood system which was named the Q-neighborhood system and used it to study convergence of sequence and nets or Moore-Smith sequence extending the study to subnets and subsequences. They also studied subspaces, separation axioms, and connectedness with the neighborhood system they have introduced, in FTSs. Hutton *et al.* [**78**] in 1980 studied hierarchy of separation axioms in FTSs and some hereditary properties of the separation axioms. Srivastava *et al.* [**172**] in 1981 studied properties of fuzzy Hausdorff spaces. Azad [**23**] in 1981 introduced fuzzy semi-open (closed), regular open (closed) sets and gave generalizations of the semi-continuous, almost continuous, and weakly continuous mappings with respect to the fuzzy sets thus defined in FTSs and observed a non-reversible one-way implication from fuzzy continuity to almost fuzzy continuity and thence to weakly fuzzy continuity. Sarkar [**145**] in 1981 studied on separation axioms and the hereditary properties and also established many results on compactness in FTSs. Studies on separation axioms of different types by various academicians and scholars in FTSs can be seen in [**63, 68, 89, 159, 186**]. Ming [**110**] in 1985 showed that all FTSs are isomorphic to certain topological spaces and added the concept of fuzzy dual points. He further introduced fuzzy metrics and studied fuzzy metrization on FTSs. Yager [**188**] in 1987 introduced fuzzy bag or multiset and used it as another approach to study the cardinality of a fuzzy set. Yalvac [**189**] studied semi-interior and semi-closure in fuzzy sets. Li [**97**] in 1990 extended the study of the cardinality of a fuzzy set and studied some applications of the fuzzy bags or multisets. Some more study in fuzzy multisets or multi-fuzzy sets can also be seen in [**148, 149**].

Studies that have been made so far on the fuzzy set have been mainly focused on the various properties of the fuzzy set as conceived by Zadeh in the year 1965 and the FTS defined by Chang in 1968 based on the fuzzy set conceptualized by Zadeh [**190, 191**]. In all these studies, the subject matter was confined only on the membership function of the fuzzy set and the related topological space. The other studies made on the topological properties with regard to the fuzzy set had also been solely on the basis of the ideas put forward by Zadeh while defining the fuzzy set for the first time. No thought was wasted to an entity that was not a member of the fuzzy set and even if anyone had thought about that nothing was available in literature till 1986 until the article by Atanassov [**20**] got published. However, before that, the conception of the intuitionistic fuzzy set (IFS) had been submitted by Atanassov in June 1983 in a "V. Sgurev Ed., VII ITKR's Session, Sofia, June 1983 (Central Sci.–Techn. Library, Bulg. Academy of Science, 1984)". This information has been collected from his published article, see references "[1]" of [**20**]. Atanassov conceptualized the IFS by introducing the degree of non-membership as another component to the fuzzy set. He also defined the interior and closure for the IFS and proved many results which are associated with the operations like containment, complementation, union, intersections etc. on sets in general. Gau *et al.* [**66**] in 1993 defined a vague set to be a set where every member had a grade of membership. However, Bustince *et al.* [**41**] in 1996 concluded that vague sets are not different from the IFSs. Coker [**45**] in 1997 introduced the intuitionistic fuzzy topological space (IFTS) and further defined continuity, compactness, connectedness and some axioms ofseparation with regard to the IFTS. Further studies of separation axioms can be seen in [**33, 34, 46, 101, 102, 156**], with regard to the IFTS. Some studies with regard to continuity and continuous functions in IFTS can be seen in [**75, 76, 103**]. Studies on fuzzy bitopological spaces and intuitionistic fuzzy bitopological spaces by various scholars can be seen in [**84, 94, 138, 173**]. More study on fuzzy interior and closure of a fuzzy set has been done in [**26, 27**].

In an effort to investigate the ambiguity and uncertainty surrounding events and collections that the classical set was unable to fully characterize, Zadeh introduced the fuzzy set in 1965. A certain amount of vagueness could be examined when the degree of membership was used as a criterion of being a member of a set in the fuzzy set instead of being fully in or fully out. This could be further improved upon by the introduction of the non-membership function by Atanassov to form the intuitionistic fuzzy set which by then had the components of membership as well as non-membership in the fuzzy set. The matter of how strongly members of the fuzzy set contained or not contained to the fuzzy set became clearer. In other words, it had become possible to determine an entity's membership or non-membership in a collection and the degree to which it belonged or not belonged to the collection. However, since membership and non-membership are degrees or grades between 0 and 1, rather than absolute numbers, it was still unclear whether the total or sum of the grades of membership and grades of non-membership added up to 1 as in the context of probability theory where the sum of the probabilities of the occurrence and non-occurrence of an event always sums up to 1. All these impending ambiguities led to the proposal of the neutrosophic logic by Smarandache [**160**] in 1998. The neutrosophic logic added a percentage of indeterminacy to the intuitionistic fuzzy set. Consequently, Smarandache [**161**] defined the neutrosophic set in 2005. Smarandache defined the neutrosophic set to encompass three components in the fuzzy set: the truth (T), the indeterminacy (I), and the falsity (F). Thus, a neutrosophic set has three functions to control the uncertainty. In the IFS defined by Atanassov [**20**], the T was taken care of by the grade of membership and F was taken care of by the grade of non-membership but the "I", the indeterminate part was lacking. In the fuzzy set initiated by Zadeh [**190, 191**], vagueness was studied by only a single membership function, degrees or grades of T of the neutrosophic set. Therefore, in order to analyze any kind of vagueness or fuzziness, it was presumed that the neutrosophic set defined by Smarandache was complete, so as to the analysis of the logic of vagueness. The neutrosophic set had been defined by Smarandache [**161**]in order to generalize the intuitionistic fuzzy set by Atanassov [**20**] which itself was an extension or improvement of the fuzzy set developed by Zadeh [**190, 191**]. The conception of the neutrosophic set had been derived by Smarandache from various real-life logics proposed by different intellects in various analysts of various real-life situations. One such logic, which has been referred to, by Smarandache, could be the four-valued logic, which had to be proposed by Belnap Jr. [**35**] in 1977, while arguing to determine the reality of a case, when four possible conclusions had to be presented before a jury, about the case: T (True), F (False), None, Both. Thus, as can be observed, each of the four possibilities could be partially true as no unitary conclusion had any proof to the particular case in context. In such a situation, the latter two components of "None" and "Both" lead to Indeterminacy. In the context of such ambiguities, the neutrosophic set, proposed by Smarandache, can be used to summarize the unknown or the indeterminate component up to a certain extent in such arguments even though it has to be agreed that in real life situations, indeterminacy worse than such cases exists.

After Salama *et al.* [**142**] in 2012 defined the neutrosophic topological space on the basis of the neutrosophic set; a lot of attention has been deviated to the study of neutrosophic topological spaces, by many topologists. This is perhaps because the neutrosophic topological spaces could be observed as aspontaneous generalization for the FTSs as stated in [**142**]. Salama *et al.* [**142**] formalized the definition of the neutrosophic set by providing proper notations and elaborated the possible arithmetic and algebraic operations between the sets and further defined the topological properties of containment, interior, and closure in neutrosophic topological spaces and established many results on these topological properties. Salama *et al*. [**143**] in 2014 redefined the neutrosophic closed set and introduced neutrosophic continuous functions and established many topological results in continuity in the neutrosophic topological space. Salama *et al.* [**144**] in 2014 defined the neutrosophic crisp set and basing on the new type set the neutrosophic crisp topological space was defined and properties of interior and closure was studied in the new type of topological space. Further, study on continuous functions in such a topological space was conducted and the concept of compactness was also extended to the neutrosophic crisp topology. Al-Omeri [**11**] in 2016 extended the study of the neutrosophic crisp topology by introducing the concepts of neutrosophic crisp α-open, β-open, semi-open, pre-open sets and studying many of the corresponding concepts like interior, closure and continuity with respect to the new open sets. Karatas *et al*. [**85**] in 2016 introduced the neutrosophic interior, neutrosophic exterior, neutrosophic closure, neutrosophic boundary and neutrosophic subspace and studied many of their topological properties. Al-Omeri [**12**] in 2016 introduced neutrosophic semi-open, pre-open, $\alpha - open$, $\beta -$ open sets and studied the corresponding continuous functions defined with respect to these open sets. Different forms of continuity of functions have been studied by various thinkers in neutrosophic topological spaces and can be observed in [**83, 93, 132**]. Studies on separation axioms in neutrosophic topological spaces have been attempted by various scholars and can be noticed in [**01, 51, 73, 88, 109**]. Multisets have been studied in neutrosophic topological spaces by some scholars and are seen in [**31, 32, 50**]. Neutrosophic study of bitopological spaces have been done by scholars in [**09, 55, 114, 125**].

After the introduction of neutrosophic logic [**160**], subsequent definition of neutrosophic set [**161**]and the introduction of the neutrosophic topological space [**142**], a lot of other studies are also seen to be done in algebraic structures with regard to the neutrosophic set. Studies in algebraic structures could also be noticed by breaking up the components of the neutrosophic sets into three parts, namely the T (Truth), I (Indeterminacy) and F (Falsity) which Smarandache [**161**] in 2005 gave the notions: <A> for T, <Neut-A> for I and <Anti-A> for F for an idea or concept "A". Study on the algebras with <A> have been classified as the classical algebras, studies on the algebras with <Neut-A> classified as NeutroAlgebras and studies on the algebras with <Anti-A> have been classified as AntiAlgebras. Similarly, studies on geometry with the three components of the neutrosophic set have been classified as Classical Geometry, NeutroGeometry, and AntiGeometry. Thus, many studies evolved outside the classical ideas with respect to the Neutro and Anti components of the neutrosophic sets. Smarandache [**162–165**], introduced NeutroAlgebra and AntiAlgebra by defining the algebraic operations of associativity, commutativity, unit element, inverse element, etc. in terms of Neutro and Anti by citing examples on each new definition by logic thereby laying the foundation for further studies in algebraic structures like groups, rings etc. He used the terms Neutrosophication and Antisophication to generalize the concepts in classical algebra to NeutroAlgebra and AntiAlgebra by defining terms like NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, NeutroOperation, AntiOperation etc. He claimed that the well-defined axioms and operations in classical algebra do not always hold in real life situations. Smarandache *et al.* [**166**] studied BE-algebras and defined neutro-BE-algebra and anti-BE-algebra. BE-algebra was first conceptualized and studied by Kim *et al.* [**60**] in order to generalize a BCK-algebra [**82**]. Further, they proposed that for any classical algebra with k operations or axioms defined on it there will be equivalently $2^k - 1$ neutro-algebras and $3^k - 2^k$ anti-algebras with respectively the same number of neutro-operations or neutro-axioms and anti-operations or anti-axioms. Agboola *et al.* [**02**] tried to realize the concept of neutro-algebra and anti-algebra [**162**] with respect to existing number systems, and found that the set of natural number system and the algebraic operation of subtraction and division form a neutro-semi group. With the same operations, the set of integers is also found out to be a neutro-semi group. It was also found that rational numbers also form a neutro-semi group with the operation of subtraction. The authors also introduced the term neutro-field and further, it was also found that rational, real and complex numbers, with the operation division, form a neutro-field. Further, Agboola [**03**] provided the formal definition of a neutro-group having associated the term neutro-semi group with the classical number systems in [**02**]. The author presented many results on neutro-group, defined neutro-subgroup, neutro-group-morphisms and established many results but however found out that some of the classical results of group theory do not hold in the setting of the neutro-group theory. Agboola [**04**] introduced the notion of neutro-rings with the help of three neutro-axioms namely, additive neutro-abelian group, the multiplicative neutro-semi group and the multiplication over addition neutro-distributivity. The 1st isomorphism theorem of the classical ring theory stood valid in the neutro-ring theory. Smarandache *et al.* [**167**] studied BCK-algebra [**82**] and extended the study to neutro-BCK-algebra by the application of neutrosophication of the underlying operations of the BCK-algebras. Agboola [**05–07**] dwelt on finite neutro-groups and finite and infinite neutro-rings. In [**07**] the author introduced anti-groups with the use of anti-operations on the operations of the classical group. Anti-operations are operations which are exactly the opposite of the classical operations. As such, closure law is characterized by anti-closure, associativity by anti-associativity and so on. Of the four anti-laws for the four axioms that are required in defining a classical group, the anti-group is defined if at least one anti-law is satisfied. Further, the author defined that an anti-group will be anti-abelian if the anti-commutativity axiom is satisfied. Agboola *et al.* [**08**] introduced anti-rings by defining ten neutro-axioms and ten anti-axioms corresponding to the ten axioms of a classical commutative ring. Each of the ten axioms included a neutro-commutativity axiom and an anti-commutativity axiom for the operation of multiplication. Then the anti-ring is defined to be a generalization of the classical ring satisfying some classical axioms and at least one of the nine anti-axioms. Further, the definition says that if the axiom of anti-commutativity of multiplication is also satisfied then the anti-ring becomes an anti-commutative ring. Surprising results have been found in the study like sub anti-rings possesses properties not common to the parent anti-ring and so on even if identical binary operations are employed. Ibrahim *et al.* [**80**] introduced neutro-vector space and studied certain simple properties of only a particular type of neutro-vector space which they called type 4*S*. Ibrahim *et al.* [**81**] defined neutro-hypergroup and anti-hypergroup by neutrosophication and antisophication of the three axioms of a classical hypergroup. They further established that for every classical hypergroup there could be possibly seven classes of neutro-hypergroups and nineteen classes of anti-hypergroups basing on the proposal given in [**166**]. Mohammadzadeh *et al.* [**112**] introduced neutro-nilpotent groups and studied some of their properties and they found the quotient of the group in context and the intersection of two such groups are also of the same group. Further, study by neutro-homeomorphism revealed that the property of neutro-nilpotency is preserved by homomorphic image. Al-Tahan *et al.* [**17**] studied the application of the new algebras to Semigroups by introducing partial order relation in the Neutro-algebras. Smarandache [**168**], diversified the study of the neutrosophic triplets Truth: <A>; Indeterminate: <Neut A>; Falsity: <Anti A> on various possible studies and analysis of space, event etc. and for any study on any structure, by neutrosophication we can always have the component neutro-structure and by antisophication, one can always have the anti-structure component. Analysis of any structure may be with anything like axiom, theorem, lemma, property, proposition etc. According to him, all of them will always be accompanied by the Neutro and the Anti components in any study or analysis. Smarandache [**169**] extended the study of neutro-algebras and anti-algebras to geometry and proposed that as non-Euclidean geometry is a result of some geometric object not following one particular postulate of Euclid, likewise not following any other postulate or all the postulates of Euclid would result in anti-geometry and not being able to conclude whether a particular object follows all the postulates of Euclid would lead to neutro-geometry. The author took the examples of Riemannian geometry and Smarandache Geometry in establishing the concept of neutro-geometry and anti-geometry and illustrated by providing complex geometric models. Rezaei *et al.* [**135**] and Al-Tahan *et al.* [**16**] extended the study of neutrosophication and antisophication to the study of Semi-hypergroups and Neutro-hyper-structures. Sahin *et al.* [**140**] in 2021 defined a neutro-metric space and discussed the basic properties of the neutro-metric, studied various similarities and differences between the neutro-metric and the classical metric and concluded that a neutro-metric space is obtainable from every classical metric space. They further studied convergence in the neutro-metric spaces by defining neutro-Cauchy sequence. Further, Sahin *et al.* [**139**] in 2021 introduced the concept of a neutro-topological space and an anti-topological space from the concept of NeutroSophication and Antisophication defined by Smarandache [**163**]. In the new topological structures that have been introduced, the neutrosophic concept is retained but the neutrosophic set has not been used. The study compared the new topologies with the general topology and concluded that neutro-topology has a more general structure than the classical topology. They also further concluded that a neutro-topology could be deduced from any given classical topology and further that a neutro-topology could also be deduced from any given anti-topology.The authors finally claimed that they have added two more structures to the study of neutro-algebra and anti-algebra.

Further, Smarandache [**170**] in 2022 made a conclusion that the NeutroAlgeras and AntiAlgebras are parts of the general forms of the classical algebras and that the classical algebra is just a part of the whole structure of algebras if to be considered from the neutrosophic point of view. Sahin [**141**] defined neutro-sigma and anti-sigma algebras by using the neutrosophication and antisophication of the classical sigma algebra and concluded that a neutro-sigma algebra can be established from every sigma algebra and a neutro-sigma algebra can also be established from every anti-sigma algebra. Witczak [**180**] studied some properties of the anti-topological space defined in [**139**] with regard to interior, closure and studied door spaces in terms of the anti-topology and also introduced two types of continuity in anti-topological spaces which are not equivalent to one another. The author also studied the concept of dense and nowhere dense sets in anti-topological spaces. Basumatary *et al.* [**30**] extended the study of topological groups to neutro-topological spaces and studied properties of group neutro-topological spaces. Witczak [**182**] defined anti-minimal topological space and the anti-bi-minimal space and studied the properties of interior and closure in the Anti set up. Anti-bi-minimal spaces being a structure associated with two anti-minimal structures. Basumatary *et al.* [**29**] studied neutro-topological neighbourhood and neutro-topological base and in [**28**] established formulae to find the number of neutro-topological spaces based on the number of members in the universal set on which a neutro-topological space is defined. Polvan [**130**] redefined the neutro-metric space defined in [**140**] by adding the properties of neutro-open sets, different from those of a neutro-topology, to define strong neutro-metric space and established that every metric space can be a strong neutro-metric space.

A neutro-topological space [**139**] is conceived as a collection of subsets of a universe in such a way that if the empty set belongs to the collection, then the universe does not belong to the collection and vice versa or altogether, it is not clear whether either of them belonged to the collection. The union and intersection of the members of the collection may or may not belong to the collection. So, it is a structure where the axioms of a GTS may or may not be completely satisfied. However, the empty set and whole set will not belong to the topology simultaneously or it may be such that they may not both belong to the topology at all since that may be indeterminable. The inclusion of the union and intersection of the members of the collection of subsets may also be at times indeterminable. However, study of such weaker structures had already been done in the past, an instance may be topologies like the supra topology [**106**], introduced in 1983, which also does not say anything about the inclusion of the empty set or the intersection of the members of the collection in the topology as it is only based on the openness the universe set and the union of the members. However, further analyses with respect to subspace, continuity, three separation axioms, closed maps, strongly closed maps had been studied by the authors in [**106**] with respect to the supra topology. The authors remarked that every continuous function is supra continuous. The authors also introduced supra interior, supra closure and supra boundary but did not study their properties in details. However, the authors remarked that the intersection of supra open sets should not be necessarily contained in the supra topology and as such the openness of the intersection of supra-open sets does not matter for a supra topological space. The authors agreed that some of the properties of the operators that are valid in topology are not valid with the supra topology. The context of referring to the supra topology in reference to the notion of a neutro topology [**139**] is the comparability of the two structures with respect to the inclusion of the intersection of members of the collection of subsets. In the neutro-topology [**139**], the intersection of the member subsets may or may not belong to the collection and the supra topology [**106**] is not at all concerned about the intersection of the subsets of the collection. Many other studies [**14, 146, 147**] are conducted by some academicians with regard to the supra topology over the years but the subject matter of the other studies may not be of further interest to the context of this study and as such detailed survey has not been provided here. Another structure which is not necessarily concerned about the union of the subsets in the collection of subsets in the topology is the infra topology [**10**], introduced in 2015. The infra-topology [**10**] or the infra-topological space is defined over a universe as a collection of subsets of the universe where the empty set is open along with the openness of the finite intersection of the collection of subsets. The arbitrary union or even finite union is not included in the definition of the infra topology [**10**]. However, the author stated that finite union of infra-open sets need not be necessarily infra-open. The author also stated that every general topology is an infra-topology which is however not the case the other way around. The author also defined infra-interior, infra-exterior, infra-closure, infra-boundary, infra-limit point, infra-derived set on the infra-open sets and infra-closed sets of the infra-topological space and studied their various properties that are valid in context of classical topologies and found that some of the results are not valid in infra-topological spaces. Further, Witczak [**181**] clarifies the definition of the infra-topological space as had been given in [**10**] which seemed to be had been misunderstood, misinterpreted and misquoted by certain authors in later published articles in context to the original definition which seemed to have some ambiguity as regard to the second axiom, which created a notion of an arbitrary collection of subsets in terms of written words and a finite collection of subsets in terms of mathematically used notations in the same axiom. Witczak clarifies that the mathematically shown finite intersection of the subsets should be taken for further study on infra-topological spaces and provides a corrected definition for the same. The author further proposed that if infinite or arbitrary intersections have to be considered at all, then the classification should be termed as Alexandrov infra-topologies. Witczak further goes on to correct the conclusion on the definition of infra-interior which stated that infra-interior is the biggest infra-open set contained in the infra-interior, which could not be true in general, as union of infra-open sets is not necessarily infra-open [**10**] in an infra-topology. Witczak provides a counter-example to illustrate the claim and the error that was inherent in the conclusion of the infra-interior of a set in infra-topological spaces. Witczak goes on to correct some more basics in the initial results and definitions of [**10**] thereby aiding in laying down a more concrete base for further studies in infra-topological spaces by including additional logical tools for the field. Al-Shami *et al.* [**13**] appreciated the improvements and corrections suggested by Witczak [**181**] and incorporated the revised concepts to their studyon separation axiomsand continuity in infra-topological spaces. The authors found that some of the properties of separation axioms are not exactly the same in infra-topological spaces but rather they behave to have other distinctive characteristics of their own. Other further studies on the infra-topological spaces can be seen in [**13, 15**]. Detailed analysis of the articles has not been done on the other studies done by various academicians in the field of infra-topology because the only purpose for mentioning infra-topological spaces in this thesis is to draw the attention to the omission of the union of open sets and to provide an example that the union of subsets of a topology may or may not belong to the topology in context, which is the case in the case of a neutro-topology where the unionand the intersection of the subsets of the universal set in context may or may not belong to the collection of subsets of the topology. Thus, it may be stated that every supra topology is a neutro topology and a neutro topology can be obtained from any infra topological space by either the exclusion of the whole universal set or the empty set. Thus, there exists a connection of the neutro-topology with other topologies already defined and studied. Another topology that will be studied is the anti-topology defined in [**139**]. Anti-topology comes from that component of the neutrosophic study which deals with the falsity of a subject under study. Since <A> and <Anti A> are the opposites of the same structure, so the anti-topology had been defined as opposite to classical definition of a topology. In other words, the axioms of an anti-topology are just the negations of the axioms of the set theoretic topology. As such, in an anti-topology, the empty set and the universal set under study will not be in the topology, union: whether finite or arbitrary, will not be there in the topology and obviously the intersection of the subsets will also not be there in the anti-topology. As such there will be nothing to compare the anti-topology with the previously mentioned topologies of supra or infra, under whatsoever conditions because in those two topologies at least the universe of study or the void set belonged to the topologies and the union and intersection of the subsets belonged to one or the other topologies separately though not together. Thus, an anti-topology is a totally different structure and will be seen to have many interesting characteristics. It has already been mentioned that other than [**139**], where the definition of the anti-topological spaces was first mentioned, Witczak [**180**] had studied some preliminary properties of the anti-topology. Basumatary *et al*. [**29, 174**] in 2023 provided the definitions of neighborhood of a point, base and sub-base in neutro-topological spaces and also in anti-topological spaces and have compared the properties of the aspects to that of the GTSs and they have come across many interesting results especially in the case of anti-topological spaces. More recently, Sravani *et al.* [**171**] added some more studies on the anti-topology by introducing and analyzing some new types of open sets and a partial order with regard to the anti-topology and established many surprising results.

The current studytakes the definition of neutro-topological space and anti-topological space that were introduced in [**139**] and proceed with studying some properties of the new structures. It has been observed from [**139**], that even though the idea of neutro-topological and anti-topological spaces have been deduced gradually from the concepts of neutrosophication and antisophication, which has its roots in the neutrosophic logic, the authors did not use the neutrosophic set as the basis for defining the two new structures. Instead, they have used the classical sets in elaborating and exemplifying the study. The current study will also continue to use only the classical sets whenever any numerical justification or elaboration is deemed necessary. The first thing that is done in the current study is to accept whatever preliminary study that has been done by the authors of [**139**] and extend the study by definingand observing the properties of the interior, the exterior, the closure, and the boundary in neutro-topological spaces and anti-topological spaces. Further, a neutro-bitopological space is defined and the properties of the interior, the exterior, the closure, and the boundary are studied. Neutro-continuity and anti-continuity of functions are defined and their properties are studied. An additional weaker form of continuity, that has been termed as weakly neutro continuous of functions has also been introduced and properties of such continuity is also studied. Separation axioms of some types are defined in terms of the Neutro and Anti set up and some hereditary properties are studied. And finally, the theory of multisets is applied to the new structures by introducing neutro-multi-topology and anti-multi-topology in terms of multisets and the properties of interior, exterior, closure and boundary are studied.

1.3 Aims and Objectives

The survey of literature from point set theoretic topological spaces and its gradual development to fuzzy, intuitionistic fuzzy, neutrosophic and then to the neutro and anti-topologies shows that very little study has been made in the field of neutro-topological space (N-TS) and anti-topological space (A-TS). Some work related to anti-topology has been done with regard to some of the preliminary properties of interior and closure.

In the same work, the concept of anti-continuity has been defined in terms of the openness of the pre-image of open sets. We will use that definition to further analyze the properties of continuity, which has not been done or seen as done in literature. Many studies have been relatively done in the algebras of neutro and anti, as had been seen in literature, such as the definition and study of neutro-groups, neutro-rings, neutro-vector spaces, neutro-field and neutro-metric spaces. Also seen in literature are the studies on the anti-counterparts of the algebras like anti-groups and anti-rings. In comparison to the neutro counterparts more studies on anti-vector spaces and anti-metric spaces remain to be studied. However, our study will not be along the line of the algebras but topological spaces. A study has been observed to have been done in group neutro-topological space and a study has also been done to find the number of neutro-topological spaces. Another study on ordered anti-topological spaces has been observed as done most recently. But studies on the interior, exterior, closure, and boundary have not been done previously in neutro-topological spaces. And even though preliminary studies of interior and closure has been done in anti-topological spaces, we will extend the study to add other properties in interior and closure in anti-topological spaces and further define exterior, and boundary in anti-topological spaces and study the corresponding properties. Continuity has been defined in anti-topological spaces without analyzing the properties of continuity in the space, we will study the properties of continuous functions in the anti-topological spaces by adopting the definition that had already been proposed. Further, we will study continuity in neutro-topological spaces by defining neutro-continuity in similar views of anti-continuity. We will also introduce a weaker form of continuity in neutro-topological spaces. Studies on the separation axioms have not been done in either the neutro-topology or the anti-topology. We will define some separation axioms in both the neutro-topological and anti-topological spaces and analyze the hereditary properties. Study of multisets have been done in fuzzy topology and in the neutrosophic topology, we will extend the classical set based neutro and anti-topologies to the theory of multisets and introduce multi-neutro topological spaces (M-N-TS) and multi-anti topological spaces (M-A-TS) and study some properties of interior, closure, exterior and boundary. Further, multi-neutro-bitopological space (M-N-B-TS) has also been defined and the properties of interior, closure, boundary are studied with the use of multiset concept. By the analysis of literature, for this research work, we have set the following objectives.

The objectives of the research work are:

(i) Observing and analysing the properties of Interior, Exterior, Closure, and Boundary points in neutro-topological spaces.

(ii) Defining neutro-bitopological spaces and observing and analysing the properties of Interior, Exterior, Closure, and Boundary points with respect to neutro-bitopological spaces.

(iii) Observing and analysing the properties of Interior, Exterior, Closure, and Boundary points in anti-topological spaces.

(iv) Study on continuous functions in neutro-topological space, by introducing the concept of neutro-continuous functions and studying the properties of continuous functions in anti-topological spaces.

(v) Study on separation axioms in neutro-topological and anti-topological spaces.

(vi) Study on multi-neutro-topological spaces, multi-neutro-bitopological spaces and multi-anti-topological spaces.

1.4 Research Methodology

In the research work, the definition of *N-TS* and *A-TS* are adopted and will be used to study the properties of the *N-TS* and *A-TS*. For the purpose of studying the properties in *N-TS* and *A-TS*, the properties of Interior, Exterior, Closure, and Boundary that are well established in *GTS* will be used and corresponding studies will be made whether those properties hold in the context of the *N-TS* and *A-TS*. Further, properties of continuous functions that are already well established in *GTS* will be observed and corresponding studies will be made to evaluate their validity in the *N-TS* and *A-TS*. Further, separation axioms that are well established in *GTS* will be considered to study the same in *N-TS* and *A-TS*.Finally, the concept of multisets will be borrowed and applied to *N-TS* and *A-TS* in analyzing some of the properties of Interior, Exterior, Closure and Boundary in *M-N-TS*, *M-N-B-TS*, and *M-A-TS*.

1.5 Importance of the Research Work

The study of topology of a set shows that a set can have multiple topologies of different types. The change in the conception of the notion of a set leads to an altogether different system of analysis. Such evolutions have already taken place with the conception of the *FS* which has led to a variety of study having many practical applications in many fields. Similar conclusions can be drawn with the generalization of the study of *FS* to the *IFS*. Further, in the case of the *NS* which had been established as a generalization of the *IFS*, the study in the field has revolutionized the study of the *FS*, leading to the expansion of analysis in real life situations, generalizing established algebras to algebras that should hold in reality. Such generalizations of the *NS* led to the evolution of the *N-TS* and *A-TS*. It has been observed from literature that whenever any new topology is defined or introduced, the preliminary studies that are seen to be done in the new topology are on the interior, closure, exterior, boundary, continuity of functions, separation axioms and other basic aspects of topological spaces. Thus, since the *N-TS* and *A-TS* have been introduced very lately and the preliminary studies had not been done by anyone, we have taken up the work in this research work. As such the preliminary study that has been undertaken in this study in the aspects of the interior, exterior, closure, boundary, continuity of functions and separation axioms in the *N-TS* and *A-TS* will be helpful in further studying the other aspects of the new topological spaces and expand the scope of analysis in the two areas.

1.6. Preliminaries

This part provides a few preliminary notions, operations, and properties those will be referred to in the subsequent chapters.

Definition 1.6.1 [**64**]

For a non-empty set $\mathcal{X}$ and a collection of subsets $\mathcal{T}$ of $\mathcal{X}$, referred to as open sets, in such a way that the following axioms are true:

(O.1) $\emptyset \in \mathcal{T}$ *and* $\mathcal{X} \in \mathcal{T}$.

(O.2) *If* $\mathcal{O}_1 \in \mathcal{T}$ *and* $\mathcal{O}_2 \in \mathcal{T}$ *then* $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}$.

(O.3) *If* $\mathcal{O}_i \in \mathcal{T}$ *for every* $i \in \mathcal{I}$, *then* $\cup \{\mathcal{O}_i : i \in \mathcal{I}\} \in \mathcal{T}$.

Then the family $\mathcal{T}$ along with thethree axioms forms a topology on $\mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{T})$ represents a topological space (TS).

The formation of the topology structure is not affected with any individual processes used in creating the family $\mathcal{T}$.

Definition 1.6.2 [**64**]

A subset $\mathcal{C}$ of a TS $\mathcal{X}$ will be called closed if the complement of $\mathcal{C}$ is a member of $\mathcal{T}$. $\mathscr{C}$ denotes the class of closed sets.

Theorem 1.6.1 [64]

The class $\mathscr{C}$ of $\mathcal{X}$ will form some topology on $\mathcal{X}$ if the statements below are true:

(i) $\emptyset, \mathcal{X} \in \mathscr{C}$.

(ii) *If $\mathcal{C}_i \in \mathscr{C}$, then for finite n, $\cup_{i=1}^{n} \mathcal{C}_i \in \mathscr{C}$.*

(iii) *If $\mathcal{C}_i \in \mathscr{C}$ for every $i \in I$, then $\cap \{\mathcal{C}_i : i \in I\} \in \mathscr{C}$.*

Definition 1.6.3 [**64**]

Let $\mathcal{A} \subseteq \mathcal{X}$. Interior of $\mathcal{A}$ is defined as:$\mathcal{A}^{int} = \cup \{\mathcal{O}: \mathcal{O} \subseteq \mathcal{A}, \mathcal{O} \in \mathcal{T}\}$. Interior of $\mathcal{A}$ is union of those subsets of $\mathcal{A}$ which are included in the topology.

The exterior of $\mathcal{A}$ is the union of all subsets of the topology that do not intersect $\mathcal{A}$:

$\mathcal{A}^{e} = \cup \{\mathcal{O}: \mathcal{O} \subseteq c\mathcal{A}, \mathcal{O} \in \mathcal{T} \}$.

Remark 1.6.1 [64]

$\mathcal{A}^{int}$ is larger than any other open set contained in $\mathcal{A}$ and $\mathcal{A}^{e}$ is larger than any other open set not intersecting $\mathcal{A}$. And, $\mathcal{A}^{ext} = (c\mathcal{A})^{i}$ and $\mathcal{A}^{int} = (c\mathcal{A})^{ext}$. $\mathcal{A}$ is open if $\mathcal{A}^{int} = \mathcal{A}$ and $\mathcal{A}$ is closed if $\mathcal{A}^{ext} = c\mathcal{A}$ and $\mathcal{A}^{int} \cap \mathcal{A}^{ext} = \emptyset$.

Theorem 1.6.2 [64]

The following are true for interior and exterior operators:

$\emptyset^{int} = \emptyset; \mathcal{X}^{int} = \mathcal{X};\ \mathcal{A}^{int} \subseteq \mathcal{A}; (\mathcal{A}^{int})^{int} = \mathcal{A}^{int}; (\mathcal{A} \cap \mathcal{B})^{int} = \mathcal{A}^{int} \cap \mathcal{B}^{int}$.

$\emptyset^{ext} = \mathcal{X}; \mathcal{X}^{ext} = \emptyset; \mathcal{A}^{ext} \subseteq c\mathcal{A};\ (\mathcal{A}^{ext})^{ext} \supseteq \mathcal{A}^{int}; (\mathcal{A} \cup \mathcal{B})^{ext} = \mathcal{A}^{ext} \cap \mathcal{B}^{ext}$.

$(\cap \mathcal{A}_i)^{int} \subseteq \cap (\mathcal{A}_i)^{int}$ *in general.*

If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^{int} \subseteq \mathcal{B}^{int}$ and $\mathcal{A}^{ext} \supseteq \mathcal{B}^{ext}$.

Definition 1.6.4 [**64**]

Closure of $\mathcal{A}$ is defined: $\mathcal{A}^{cl} = \cap \{\mathcal{C}: \mathcal{A} \subseteq \mathcal{C}, c\mathcal{C} \in \mathcal{T}\}$ where $\mathcal{A} \subseteq (\mathcal{X}, \mathcal{T})$.

Theorem 1.6.3 [64]

A set $\mathcal{A}$ is closed if $\mathcal{A}^{cl} = \mathcal{A}$ and vice versa.

Also, $\emptyset^{cl} = \emptyset$; $\mathcal{X}^{cl} = \mathcal{X}$; $\mathcal{A} \subseteq \mathcal{A}^{cl}$; $(\mathcal{A}^{cl})^{cl} = \mathcal{A}^{cl}$; $(\mathcal{A} \cup \mathcal{B})^{cl} = \mathcal{A}^{cl} \cup \mathcal{B}^{cl}$;
If $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathcal{A}^{cl} \subseteq \mathcal{B}^{cl}$.

Theorem 1.6.4 [**59**]

Boundary of $\mathcal{A}$ *is defined*: $\mathcal{A}^{bd} = \mathcal{A}^{cl} \cap (c\mathcal{A})^{cl}$ *where* $\mathcal{A} \subseteq \mathcal{X}$.
Also $\mathcal{A}^{cl} \cap (c\mathcal{A})^{cl} = \mathcal{A}^{cl} \setminus \mathcal{A}^{int}$; $(c\mathcal{A})^{bd} = \mathcal{A}^{int} \cup (c\mathcal{A})^{int}$; $\mathcal{A}^{cl} = \mathcal{A}^{int} \cup \mathcal{A}^{bd}$;
$\mathcal{A}^{int} = \mathcal{A} \setminus \mathcal{A}^{bd}$.

Remark 1.6.2 [**64**]

The following holds for any $\mathcal{A}, \mathcal{B} \subseteq (\mathcal{X}, \mathcal{T})$:
$(\mathcal{A}^{bd})^{bd} \subseteq \mathcal{A}^{bd}$; $(\mathcal{A} \cap \mathcal{B})^{bd} \subseteq \mathcal{A}^{bd} \cup \mathcal{B}^{bd}$; $(\mathcal{A} \cup \mathcal{B})^{bd} \subseteq \mathcal{A}^{bd} \cup \mathcal{B}^{bd}$;
$(\mathcal{A}^{cl})^{bd} \subseteq \mathcal{A}^{bd}$; $(\mathcal{A}^{int})^{bd} \subseteq \mathcal{A}^{bd}$.

Theorem 1.6.5 [**64**]

A set $\mathcal{A}$ *is open if* $\mathcal{A}^{bd} \subseteq c\mathcal{A}$ *and closed if and only if* $\mathcal{A}^{bd} \subseteq \mathcal{A}$.

Remark 1.6.3 [**59**]

For $\mathcal{A} \subseteq (\mathcal{X}, \mathcal{T})$ *the following are true*:

(i) $\mathcal{A}^{bd} = \mathcal{A}^{cl} \setminus \mathcal{A}^{int}$

(ii) $\mathcal{A}^{bd} \cap \mathcal{A}^{int} = \emptyset$

(iii) $\mathcal{A}^{cl} = \mathcal{A}^{int} \cup \mathcal{A}^{bd}$

Definition 1.6.5 [**59**]

$\mathcal{D} \subset \mathcal{X}$ *is dense in* $\mathcal{X}$ *if* $\mathcal{D}^{cl} = \mathcal{X}$.

Remark 1.6.4 [**59**]

The statements below are equivalent:

(i) $\mathcal{D}$ *is dense in* $\mathcal{X}$.

(ii) $\mathcal{C}$ *closed and* $\mathcal{D} \subset \mathcal{C}$, *then* $\mathcal{C} = \mathcal{X}$.

(iii) *Each non-void basic open set in* $\mathcal{X}$ *has an entity of* $\mathcal{D}$.

(iv) *The complement of* $\mathcal{D}$ *has an empty interior.*

Definition 1.6.6 [**64**]

If $\mathscr{B}$, *a class of* $\mathcal{B}_i \subseteq \mathcal{X}$ *is an open basis for some* $\mathcal{T}$ *on* $\mathcal{X}$ *then:*

(i) *For all* $x \in \mathcal{X}$, $\exists$ *some* $\mathcal{B} \in \mathscr{B}$ *for which* $x \in \mathcal{B}$.

(ii) *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are members of the class $\mathscr{B}$ with x contained in their intersection, then $\exists\ \mathcal{B}$ in $\mathscr{B}$ for which $x \in \mathcal{B} \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$.*

The members of $\mathcal{T}$ are union of some members of $\mathscr{B}$.

Definition 1.6.7 [59]

For a TS $(\mathcal{X}, \mathcal{T})$ and $\mathcal{A} \subset \mathcal{X}$, the induced topology $\mathcal{T}_{\mathcal{A}}$ on $\mathcal{A}$ is $\{\mathcal{A} \cap \mathcal{U} | \mathcal{U} \in \mathcal{T}\}$ and is called subspace of the TS $\mathcal{X}$.

Definition 1.6.8 [113]

For two TS $\mathcal{X}$ and $\mathcal{Y}$, a mapping f from $\mathcal{X}$ to $\mathcal{Y}$ will be called continuous if for every open set $\mathcal{W}$ in $\mathcal{Y}$, the set $f^{-1}(\mathcal{W})$ is open in $\mathcal{X}$.

Theorem 1.6.6 [113]

For the TS $\mathcal{X}$ and $\mathcal{Y}$, and a mapping f from $\mathcal{X}$ to $\mathcal{Y}$, the statements below are equivalent:

(i) f is a continuous function.

(ii) For every closed subset $\mathcal{B}$ of $\mathcal{Y}$, $f^{-1}(\mathcal{B})$ is a closed subset of $\mathcal{X}$.

(iii) For each subset $\mathcal{A}$ of $\mathcal{X}$, $f(\mathcal{A}^{cl}) \subset (f(\mathcal{A}))^{cl}$.

(iv) For all $x \in \mathcal{X}$ and open set $\mathcal{W}$ that contain $f(x)$, an open set $\mathcal{V}$ that contain x exists, so that $f(\mathcal{V}) \subset \mathcal{W}$.

Theorem 1.6.7 [113]

Consider the TSs $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, then the following statements are always true.

(i) When a mapping f from $\mathcal{X}$ to $\mathcal{Y}$, maps the whole of $\mathcal{X}$ into a single value a of $\mathcal{Y}$, then f is continuous.

(ii) When $\mathcal{A}$ is a subspace for $\mathcal{X}$ then the function f from $\mathcal{A}$ to $\mathcal{X}$ is continuous.

(iii) When a mapping f from $\mathcal{X}$ to $\mathcal{Y}$ and a mapping g from $\mathcal{Y}$ to $\mathcal{Z}$ are both continuous then composite of f and g represented by $g \circ f$ from $\mathcal{X}$ to $\mathcal{Z}$ is also continuous.

(iv) When a function f from $\mathcal{X}$ to $\mathcal{Y}$ is continuous, with $\mathcal{A}$ a subspace of $\mathcal{X}$, then $f|\mathcal{A}$, the function whose domain is restricted to $\mathcal{A}$, the mapping from $\mathcal{A}$ to $\mathcal{Y}$, is continuous.

(v) When a mapping f from $\mathcal{X}$ to $\mathcal{Y}$, is continuous, and $\mathcal{Z}$, a subspace of $\mathcal{Y}$ that contains the image set $f(\mathcal{X})$, then the function g from $\mathcal{X}$ to $\mathcal{Z}$ obtained by

restricting the domain of $\mathcal{f}$, is continuous. If $\mathcal{Z}$ is a space having $\mathcal{Y}$ as a subspace, then a map $\mathcal{f}$ from $\mathcal{X}$ to $\mathcal{Z}$ obtained by enlarging the range of $\mathcal{f}$, is continuous.

Theorem 1.6.8 [64]

If $\mathcal{f}$, a map of $\mathcal{X}$ into $\mathcal{Y}$ is continuous, then for all $\mathcal{W}$ open in $\mathcal{Y}$ the following are true:

(i) $(\mathcal{f}^{-1}(\mathcal{W}))^{cl} \subseteq \mathcal{f}^{-1}(\mathcal{W}^{cl})$

(ii) $\mathcal{f}^{-1}(\mathcal{W}^{int}) \subseteq (\mathcal{f}^{-1}(\mathcal{W}))^{int}$

Definition 1.6.9 [64]

A TS $\mathcal{X}$ is T_0 space if $\forall\, a \neq b$ in $\mathcal{X}$ $\exists$ open set $\mathcal{O}$ with: $a \in \mathcal{O}$ but $b \notin \mathcal{O}$ or $b \in \mathcal{O}$ but $a \notin \mathcal{O}$. Put otherwise, $\mathcal{O}$ contains one of x and y but not the other.

Definition 1.6.10 [64]

A TS $\mathcal{X}$ is T_1space if $\forall a \neq b$ in $\mathcal{X}$ $\exists$ open sets $\mathcal{O}_a$ and $\mathcal{O}_b$ so that $a \in \mathcal{O}_a, b \notin \mathcal{O}_a$ and $b \in \mathcal{O}_b$, $a \notin \mathcal{O}_b$.

Remark 1.6.5 [64]

Every space that satisfies T_1 axiom satisfies the T_0 axiom.

Theorem 1.6.9 [64]

A TS $\mathcal{X}$ is a T_1 space if each of the statements below are true:

- *(i)* *Every singleton set $\{a\}$ is closed.*
- *(ii)* *For any $\mathcal{A}$ and some open $\mathcal{B}$ so that $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{O}_{\mathcal{A}} \cap \mathcal{B} = \mathcal{A}$.*
- *(iii)* *If $a \in \mathcal{O}_i$ for all i, then$\cap\, \mathcal{O}_i = \{a\}$.*
- *(iv)* *If $\mathcal{A} \subseteq \mathcal{X}$, then $\mathcal{A} = \cup \{\mathcal{C}: \mathcal{C}\ is\ closed\ in\ \mathcal{X}\}$.*
- *(v)* *If $\mathcal{A} \neq \emptyset$, then $\mathcal{A} \supseteq \mathcal{C} \neq \emptyset$ and $\mathcal{C}$ is closed in $\mathcal{X}$.*

Definition 1.6.11 [64]

A TS $\mathcal{X}$ is T_2 or Hausdorff space if $\forall a \neq b$ in $\mathcal{X}$ $\exists$ open sets $\mathcal{O}_a$ and $\mathcal{O}_b$ such that $\mathcal{O}_a \cap \mathcal{O}_b = \emptyset$ and $a \in \mathcal{O}_a$ and $b \in \mathcal{O}_b$.

Remark 1.6.6 [64]

Every T_2 space is a T_1 space.

Remark 1.6.7 [**64**]

*All the properties in **Theorem 1.6.9** holds true for a T_2 space because of **Remark 1.6.5**.*

Theorem 1.6.10 [**64**]

A TS is a T_2 if intersection of closed sets that contain the point a is the set $\{a\}$.

Definition 1.6.12 [**64**]

A TS is called a T_3 space when for a closed $\mathcal{A}$ and a point $\mathcal{b} \notin \mathcal{A}$ there exists open sets $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{b}}$ so that $\mathcal{O}_{\mathcal{A}} \cap \mathcal{O}_{\mathcal{b}} = \emptyset$ and $\mathcal{A} \subseteq \mathcal{O}_{\mathcal{A}}$ and $\mathcal{b} \in \mathcal{O}_{\mathcal{b}}$.

Theorem 1.6.11 [**64**]

If a TS satisfies T_0 and T_3 axioms then it is a Hausdorff space

Theorem 1.6.12 [**64**]

A TS $\mathcal{X}$ is a T_3 space if the following holds in $\mathcal{X}$:

(i) *For any non-void open $\mathcal{O}$ and any $x \in \mathcal{O}$, $\exists$ open $\mathcal{O}_x$ so that $x \in \mathcal{O}_x \subseteq {\mathcal{O}_x}^{cl} \subseteq \mathcal{O}$.*

(ii) *For all closed $\mathcal{A}$, intersection of closed sets that contains the set is the set itself.*

(iii) *For any $\mathcal{A}$ and a set $\mathcal{B}$, which is open and satisfies $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ $\exists$ $\mathcal{O}$, which is open so that $\mathcal{A} \cap \mathcal{O} \neq \emptyset$ and $\mathcal{O}^{cl} \subseteq \mathcal{B}$.*

(iv) *For every non-void $\mathcal{A}$ and closed $\mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\exists$ open $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ with $\mathcal{O}_{\mathcal{A}} \cap \mathcal{O}_{\mathcal{B}} = \emptyset$ so that $\mathcal{A} \cap \mathcal{O}_{\mathcal{A}} \neq \emptyset$ with $\mathcal{B} \subseteq \mathcal{O}_{\mathcal{B}}$.*

Definition 1.6.13 [**64**]

A TS $\mathcal{X}$ will be classified as a T_4 space if for closed sets $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\exists$ open sets $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ with $\mathcal{O}_{\mathcal{A}} \cap \mathcal{O}_{\mathcal{B}} = \emptyset$, such that $\mathcal{A} \subseteq \mathcal{O}_{\mathcal{A}}$ and $\mathcal{B} \subseteq \mathcal{O}_{\mathcal{B}}$.

Theorem 1.6.13 [**64**]

If a TS satisfies the T_1 and T_4 axioms, then it also satisfies T_2 and T_3 axioms.

Theorem 1.6.14 [**64**]

A TS is T_4 if and only if for all open set $\mathcal{O}$ and all closed $\mathcal{A} \subseteq \mathcal{O}$ there is an open $\mathcal{O}_{\mathcal{A}}$ so that $\mathcal{A} \subseteq \mathcal{O}_{\mathcal{A}} \subseteq {\mathcal{O}_{\mathcal{A}}}^{cl} \subseteq \mathcal{O}$.

Definition 1.6.14 [**190**]

A mapping $\mathcal{A}: \mathcal{X} \to [0,1]$ *defined on* $\mathcal{X}$ *defines a fuzzy set over* $\mathcal{X}$, *and the function* $\mathcal{A}$ *is termed as a membership function and* $\mu_{\mathcal{A}}(x)$ *is membership grade for the member* x.

A fuzzy set is also represented as $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) : x \in \mathcal{X}\}$, *with every pair* $(x, \mu_{\mathcal{A}}(x))$ *being called a singleton.*

Definition 1.6.15 [**44**]

A collection of fuzzy subsets $\mathcal{T}$ *of* $\mathcal{X}$ *called fuzzy open, is called a fuzzy topology (FT) on* $\mathcal{X}$ *if the following three axioms are true*:

(i) $0_{\mathcal{F}}, 1_{\mathcal{F}} \in \mathcal{T}$.

(ii) $\cup_{i \in \mathcal{I}} \mathcal{U}_i \in \mathcal{T}$ *of any class* $\{\mathcal{U}_i : i \in \mathcal{I}\}$.

(iii) *For* $\mathcal{U}_1$ *and* $\mathcal{U}_2 \in \mathcal{T}$, *the fuzzy set* $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{T}$.

Then $(\mathcal{X}, \mathcal{T})$ *is termed as a FTS. The very existence of* $\mathcal{T}$ *in* $[0,1]$ *necessarily implies that* $0_{\mathcal{F}}$ *and* $1_{\mathcal{F}}$ *are both open in* $\mathcal{T}$.

Definition 1.6.16 [**84**]

A fuzzy bitopological space is a triplet $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, *where* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are FTs on* $\mathcal{X}$.

Definition 1.6.17 [**161**]

If $\mathcal{T}, \mathcal{I}, \mathcal{F}$ *are real subsets of* $]0^-, 1^+[$, *with:*

n_supremum = sup $\mathcal{T}$ *+sup* $\mathcal{I}$ *+sup* $\mathcal{F}$ *and n_infimum = inf* $\mathcal{T}$ *+inf* $\mathcal{I}$ *+inf* $\mathcal{F}$, *where* $\mathcal{T}, \mathcal{I}, \mathcal{F}$ *are called the neutrosophic components.*

If $\mathcal{A}$ *is a set contained in* $\mathcal{X}$, *then* $x \in \mathcal{X}$ *is identified with regard to the set* $\mathcal{A}$ *as a value associated with the triplet:* $x\langle \mathcal{T}, \mathcal{I}, \mathcal{F} \rangle$ *and will be associated with* $\mathcal{A}$ *as follows: it will be t% true in* $\mathcal{A}$, *i% indeterminable (unknown) in* $\mathcal{A}$, *and f% false, where* t *ranges in* $\mathcal{T}$, i *ranges in* $\mathcal{I}$, f *ranges in* $\mathcal{F}$.

[**142**]*In other words,* $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}, \sigma_{\mathcal{A}}, \gamma_{\mathcal{A}} \rangle : x \in \mathcal{X}\}$, *where* $\mathcal{T}, \mathcal{I}, \mathcal{F} : \mathcal{X} \to [0,1]$ *satisfying the criteria:* $0 \le \mu_{\mathcal{A}} + \sigma_{\mathcal{A}} + \gamma_{\mathcal{A}} \le 3$ *and* $\mu_{\mathcal{A}}(x), \sigma_{\mathcal{A}}(x)$, *and* $\gamma_{\mathcal{A}}(x)$ *stands for the membership grade, indeterminacy grade and the non-membership grade respectively for each member* $x \in \mathcal{X}$ *in the set* $\mathcal{A}$. *Other ways of expressing the set* $\mathcal{A}$ *are:* $\mathcal{A} = \langle \frac{x}{(T(x), I(x), F(x))} : x \in \mathcal{X} \rangle$ *or as* $\mathcal{A} = \langle \frac{x}{(\mu_{\mathcal{A}}(x), \sigma_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x))} : x \in \mathcal{X} \rangle$.

Definition 1.6.18 [**142**]

For a non-empty set $\mathcal{X}$ *and a collection* $\mathcal{T}_N$ *of neutrosophic subsets in* $\mathcal{X}$ *if*:

(i) $0_N, 1_N \in \mathcal{T}_N$.

(ii) $\mathcal{A}_1 \cap \mathcal{A}_2 \in \mathcal{T}_N$ *whenever* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are members of* $\mathcal{T}_N$.

(iii) $\cup \mathcal{A}_i \in \mathcal{T}_N$ *for any arbitrary collection* $\{\mathcal{A}_i : i \in \mathcal{I}\} \in \mathcal{T}_N$.

Then $\mathcal{T}$ *is a neutrosophic topology for* $\mathcal{X}$ *and the pair* $(\mathcal{X}, \mathcal{T}_N)$ *becomes neutrosophic topological space and members of* $\mathcal{T}_N$ *are neutrosophic open sets.*

Remark 1.6.8 [**139**]

The symbols " $=_I$ " *and* " $\in_I$ " *are used to denote respectively circumstances when "equal to" and "member of" are not sure or not properly defined.*

Definition 1.6.19 [**139**]

For a non-empty set $\mathcal{X}$, *and a class* $\mathcal{T}$ *of subsets of* $\mathcal{X}$, *if one or more of {i, ii, iii} below are true, then* $\mathcal{T}$ *becomes a neutro-topology (N-T) on* $\mathcal{X}$ *and the pair* $(\mathcal{X}, \mathcal{T})$ *will becalled a neutro-topological space (N-TS).*

(i) $[\emptyset \text{ and } \mathcal{X} \notin \mathcal{T} \text{ simultaneously}]$ *or* $[\emptyset, \mathcal{X} \in_I \mathcal{T}]$

(ii) *For* $p_\alpha \in \mathcal{T}$; $for\ finite\ n$, $\cap_{\alpha=1}^{n} p_\alpha \in \mathcal{T}$ *and for other* $q_\alpha, r_\alpha \in \mathcal{T}$, *for finite* n; $[\cap_{\alpha=1}^{n} q_\alpha \notin \mathcal{T}$ *or* $(\cap_{\alpha=1}^{n} r_\alpha \in_I \mathcal{T})]$

(iii) *For* $p_\alpha \in \mathcal{T}$, $\cup_{\alpha \in I} p_\alpha \in \mathcal{T}$, I *being an arbitrary index set, and for other* $q_\alpha, r_\alpha \in \mathcal{T}$, $[\cup_{\alpha \in I} q_\alpha \notin \mathcal{T}$ *or* $(\cup_{\alpha \in I} r_\alpha \in_I \mathcal{T})]$.

Remark 1.6.9 [**139**]

If none of {i, ii, iii} above are true then a N-TS follows the axioms of GTS and as such N-TS have wider scope in terms of structure than GTS.

Theorem 1.6.15 [**139**]

If $\mathcal{T}$ *is a classical topology on* $\mathcal{X}$, *then,* $\mathcal{T} \setminus \emptyset$ *is a N-T on* $\mathcal{X}$.

Theorem 1.6.16 [**139**]

If $\mathcal{T}$ *is a classical topology on* $\mathcal{X}$, *then,* $\mathcal{T} \setminus \mathcal{X}$ *is a N-T on* $\mathcal{X}$.

Remark 1.6.10 [**139**]

From the above two theorems it can be concluded that a N-T is deducible from any general topology.

Definition 1.6.20 [**139**]

For a non-void set $\mathcal{X}$ *and* $\mathcal{T}$, *a class of subsets of* $\mathcal{X}$, *if the conditions {i, ii, iii} are true, then* $\mathcal{T}$ *becomes an anti-topology (A-T) on the set* $\mathcal{X}$ *and* $(\mathcal{X}, \mathcal{T})$ *becomes an anti-topological space (A-TS).*

(i) $\emptyset, \mathcal{X} \notin \mathcal{T}$

(ii) *For all* $q_1, q_2, q_3, \ldots, q_n \in \mathcal{T}$, $(\cap_{i=1}^{n} q_i \notin \mathcal{T})$, *n being finite.*

(iii) *For all* $q_1, q_2, q_3, \ldots, q_n \in \mathcal{T}$, $(\cup_{i \in I} q_i \notin \mathcal{T})$.

Theorem 1.6.17 [**139**]

For every A-T $\mathcal{T}$ *on a set* $\mathcal{X}$, $\mathcal{T} \cup \emptyset$ *is a N-T on the set* $\mathcal{X}$.

Theorem 1.6.18 [**139**]

For every A-T $\mathcal{T}$ *on a set* $\mathcal{X}$, $\mathcal{T} \cup \mathcal{X}$ *is a N-T on the set* $\mathcal{X}$.

Definition 1.6.21 [174]

For a N-TS $(\mathcal{X}, \mathcal{T})$, *if* $x \in \mathcal{X}$, *a subset* $\mathcal{O}$ *of* $\mathcal{X}$ *is called a neutro-neighborhood of* x *iff there is a neutro-open (N-O) set* $\mathcal{M}$ *satisfying* $x \in \mathcal{M} \subseteq \mathcal{O}$.

Definition 1.6.22 [174]

For a N-TS $(\mathcal{X}, \mathcal{T})$, *a non-empty sub-collection* $\mathcal{B}$ *of subsets of* $\mathcal{X}$ *will be called a neutro-base for some N-T, provided the following conditions are true:*

(i) *If* $\mathcal{B} \subseteq \mathcal{T}$

(ii) *For each point* $x \in \mathcal{X}$ *and each neutro-nbd* $\mathcal{O}$ *of* x *there exists some* $B \in \mathcal{B}$ *so that* $x \in B \subseteq \mathcal{O}$.

(iii) *If* $\mathcal{A} \in \mathcal{T}$, *then* $\mathcal{A} = \cup \{B : B \in \mathcal{B}; B \subseteq \mathcal{A}\}$

Definition 1.6.23 [174]

For a N-TS $(\mathcal{X}, \mathcal{T})$, *if* $x \in \mathcal{X}$, *a collection* $\mathcal{B}^*$ *of subsets of* $\mathcal{X}$ *will be called a neutro-sub-base for the N-T* $\mathcal{T}$, *if* $\mathcal{B}^* \subset \mathcal{T}$ *and finite intersections of the members of* $\mathcal{B}^*$ *forms a neutro-base for* $\mathcal{T}$.

Definition 1.6.24 [180]

For two A-TSs $(\mathcal{X}, \mathcal{T}_1)$ *and* $(\mathcal{Y}, \mathcal{T}_2)$, *a mapping* f *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is said to be anti-continuous if and only if for any* $\mathcal{Q} \in \mathcal{T}_2$, $f^{-1}(\mathcal{Q}) \in \mathcal{T}_1$.

Definition 1.6.25 [180]

For two A-TSs $(\mathcal{X},\mathcal{T}_1)$ and $(\mathcal{Y},\mathcal{T}_2)$, a mapping f from $\mathcal{X}$ to $\mathcal{Y}$ is said to be anti-continuous if and only if for any $\mathcal{T}_2$-A-CS $\mathcal{C}$, $f^{-1}(\mathcal{C})$ is $\mathcal{T}_1$-A-CS.

Definition 1.6.26 [70]

A multiset $\mathcal{M}$, written in short as m-set or mset, formed from the elements of a set $\mathcal{X}$ is defined as a function count M or, alternately $C_M: \mathcal{X} \to N$; here N is the set of natural numbers. If $\mathcal{X} = \{x_1, x_2, x_3, \dots, x_n\}$ then $C_M(x)$ denotes the multiples of $x \in \mathcal{X}$ that appear in $\mathcal{M}$ and is written as $\mathcal{M} = \{\frac{m_1}{x_1}, \frac{m_2}{x_2}, \frac{m_3}{x_3}, \dots, \frac{m_n}{x_n}\}$ where m_i denotes the multiplicity of occurrence of x_i in $\mathcal{M}$. Elements of $\mathcal{X}$ that do not appear in the m-set M have count zero.

Operations on multisets $\mathcal{A}$ and $\mathcal{B}$:

(i) $\mathcal{A} = \mathcal{B}$ *if and only if* $\mathcal{C}_{\mathcal{A}}(x) = \mathcal{C}_{\mathcal{B}}(x)$ *for every* $x \in \mathcal{X}$.

(ii) $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ *iff* $\mathcal{C}_D(x) = maximum\{\mathcal{C}_{\mathcal{A}}(x), \mathcal{C}_{\mathcal{B}}(x)\}$ *for every* $x \in \mathcal{X}$.

(iii) $\mathcal{D} = \mathcal{A} \cap \mathcal{B}$ *iff* $\mathcal{C}_{\mathcal{D}}(x) = minimum\{\mathcal{C}_{\mathcal{A}}(x), \mathcal{C}_{\mathcal{B}}(x)\}$ *for every* $x \in \mathcal{X}$.

Definition 1.6.27 [70]

$\mathcal{M}^*$stands for the root set of $\mathcal{M}$, and is asubset of $\mathcal{X}$ and is defined asby$\mathcal{M}^* = \{x \in \mathcal{X}: C_{\mathcal{M}}(x) > 0\}$. It is also called the support set.

Definition 1.6.28 [70]

An empty m-set is defined as $C_{\mathcal{M}}(x) = 0$, for every $x \in \mathcal{X}$and is denoted by $\emptyset$.

Definition 1.6.29 [70]

An m-set space $[\mathcal{X}]^n$contains all m-sets having members from $\mathcal{X}$ in such a way that all members appear at most n times.

Definition 1.6.30 [70]

An mset $\mathcal{B}$ is termed a submset of $\mathcal{A}$ iff $C_{\mathcal{B}}(x) \leq C_{\mathcal{A}}(x)$ for all $x \in \mathcal{X}$.

Definition 1.6.31 [70]

Let $\mathcal{A} \in [\mathcal{X}]^n$ be an m-set, then we have the following definitions:

i. *$P(\mathcal{A})$ is the power mset of $\mathcal{A}$, and all submsets of $\mathcal{A}$ belongs to $P(\mathcal{A})$.*
ii. *$P^*(\mathcal{A})$ is the power mset of the support set of $P(\mathcal{A})$.*

Definition 1.6.32 [70]

Let $\mathcal{M} \in [\mathcal{X}]^n$ *and* $\mathcal{T}$ *is a sub-class of* $P^*(\mathcal{M})$*, then* $\mathcal{T}$ *will be a multi-topology or m-topology if:*

i. $\mathcal{M}$ *and* $\emptyset$ *are in* $\mathcal{T}$.

ii. *Arbitrary union of classes of the msets of* $\mathcal{T}$ *belongs to* $\mathcal{T}$.

iii. *Finite intersection of sub-classes of msets of* $\mathcal{T}$ *will be in* $\mathcal{T}$.

The members of $\mathcal{T}$ *are termed open msets with the corresponding complements being termed closed msets and the m-topological space will be written in short as MTS.*

Definition 1.6.33 [70]

The m-complement of a submset $\mathcal{N}$ *in an MTS* $(\mathcal{M}, \mathcal{T})$ *is denoted and defined as* $\mathcal{N}^c = \mathcal{M} \ominus \mathcal{N}$,where $\mathcal{M} \ominus \mathcal{N} = max\ \{C_{\mathcal{M}}(x) - C_{\mathcal{N}}(x), 0\}$.

Definition 1.6.34 [70]

Given a submset $\mathcal{A}$ *of an MTS* $\mathcal{M}$ *in* $[X]^n$*, then*

i. *The interior of* $\mathcal{A}$ *is defined as:* $C_{\mathcal{A}^{int}}(x) = C_{\cup\mathcal{B}}(x)$*, where* $\mathcal{B}$ *is open and* $\mathcal{B} \subseteq \mathcal{A}$.

ii. *The closure of* $\mathcal{A}$ *is defined as:* $C_{\mathcal{A}^{cl}}(x) = C_{\cap\mathcal{D}}(x)$*, where* $\mathcal{D}$ *is closed and* $\mathcal{A} \subseteq \mathcal{D}$.

Definition 1.6.35 [49]

Given a submset $\mathcal{A}$ *of an MTS* $\mathcal{M}$ *in* $[\mathcal{X}]^n$*, then*

i. *The exterior of* $\mathcal{A}$ *is given by:* $C_{\mathcal{A}^{ext}}(x) = C_{(c\mathcal{A})^{int}}(x)$ *for any* $x \in \mathcal{X}$.

ii. *The boundary of* $\mathcal{A}$ *is given by:* $C_{\mathcal{A}^{bd}}(x) = C_{\mathcal{A}^{cl} \cap\ (c\mathcal{A})^{cl}}(x)\ \forall x \in \mathcal{X}$.