CHAPTER 2

Properties of Interior, Exterior, Closure, and Boundary in Neutro-Topological Spaces

In this chapter the notions of interior, exterior, closure, and boundary are defined in neutro-topological spaces (N-TS) and the various properties of these aspects that are generally true for GTS are inspected. Whenever certain existing properties deviates from the classical properties, conditions for their validation are provided to justify the deviations in the properties or necessary counter examples are provided to justify the claims.

Proposition 2.0.1

For a non-empty set $\mathcal{X}$ and a collection $\mathcal{T}$ of subsets of $\mathcal{X}$, called neutro-open set (N-OS), $\mathcal{T}$ is a neutro-topology (N-T) and $(\mathcal{X}, \mathcal{T})$ is a neutro-topological space (N-TS) if any one of the following is satisfied:

(i) *The null-set ($\emptyset$) or the whole set ($\mathcal{X}$) is not in the collection $\mathcal{T}$.*

(ii) *There exist some members of $\mathcal{T}$ whose union does not belong to $\mathcal{T}$.*

(iii) *There exist some members of $\mathcal{T}$ whose intersection is not a member of $\mathcal{T}$.*

Example 2.0.1

If $\mathcal{X} = \{\mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s}\}$ with $\mathcal{T} = \{\{\mathrm{p}\}, \{\mathrm{s}\}, \{\mathrm{p}, \mathrm{q}\}, \{\mathrm{q}, \mathrm{r}\}, \{\mathrm{p}, \mathrm{q}, \mathrm{r}\}, \mathcal{X}\}$. Here $(\mathcal{X}, \mathcal{T})$ is a *N-TS* as can be seen from below:

(i) The null set is not *N-O.*

(ii) $\{\mathrm{p}\}, \{\mathrm{s}\} \in \mathcal{X}$ but $\{\mathrm{p}\} \cup \{\mathrm{s}\} = \{\mathrm{p}, \mathrm{s}\} \notin \mathcal{X}$.

(iii) $\{\mathrm{p}, \mathrm{q}\}, \{\mathrm{q}, \mathrm{r}\} \in \mathcal{T}$ but $\{\mathrm{p}, \mathrm{q}\} \cap \{\mathrm{q}, \mathrm{r}\} = \{\mathrm{q}\} \notin \mathcal{T}$.

Remark 2.0.1

The union of *N-Ts* need not always be a *N-T*. It can be seen by the example that follows:

Some of the results discussed in this chapter have been published in: Basumatary, B., Khaklary, J.K., Wary, N., & Smarandache, F. (2022). On Neutro-Topological Spaces and Their Properties. *Theory and Applications of NeutroAlgebras as Generalisations of Classical Algebras (pp. 180-201). IGI Global.*

Assume $\mathcal{X} = \{1,2,3,4\}$ and consider $\mathcal{T}_1 = \{\{1\},\{4\},\{1,2\},\{2,3\},\{1,2,3\},\mathcal{X}\}$ and $\mathcal{T}_2 = \{\emptyset,\{2\},\{3\},\{1,3\},\{1,4\},\{2,3,4\}\}$ to be two *N-Ts* on the set $\mathcal{X}$.
Then, $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{1,2,3\},\{2,3,4\},\mathcal{X}\}$ is not a *N-T* on $\mathcal{X}$ because both $\emptyset$ and $\mathcal{X}$ simultaneously belong to $\mathcal{T}_1 \cup \mathcal{T}_2$ and it contradicts *(i)* of the ***proposition 2.0.1***.

2.1 Interior in Neutro-Topological Spaces

Definition 2.1.1

Let $(\mathcal{X},\mathcal{T})$ be a N-TS on the set $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}$, then the neutro-interior (Nu-interior) of $\mathcal{A}$ is defined as the union of the subsets of $\mathcal{A}$ which are N-O and it is denoted by $\mathcal{A}^{Nu-int}$. That is, $\mathcal{A}^{Nu-int} = \cup\{\mathcal{U}_i : \mathcal{U}_i \subseteq \mathcal{A} \text{ and each } \mathcal{U}_i \text{ is } N\text{-}OS\}$.

Proposition 2.1.1

If $\mathcal{A}$ is a N-OS then $\mathcal{A}^{Nu-int} = \mathcal{A}$.

Remark 2.1.1

In a *N-TS* $\mathcal{X}$ and for any $\mathcal{A} \subset \mathcal{X}$, $\mathcal{A}^{Nu-int}$ need not be the largest *N-OS* contained in $\mathcal{A}$. This fact is because, in a *N-TS*, the union of *N-OS* need not be necessarily *N-O*. The following example may be considered:
Consider $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\{1\},\{4\},\{1,2\},\{2,3\},\{1,2,3\},\mathcal{X}\}$ and consider $\mathcal{A} = \{2,3,4\}$ then $\mathcal{A}^{Nu-int} = \{4\} \cup \{2,3\} = \{2,3,4\} = \mathcal{A}$. However, $\mathcal{A}$ is itself not an *N-OS*.
The above example also reveals that the converse of ***proposition 2.1.1*** is not true. That is, if $\mathcal{A}^{Nu-int} = \mathcal{A}$, then $\mathcal{A}$ need not be necessarily *N-O*.

Proposition 2.1.2

Let $(\mathcal{X},\mathcal{T})$ be a N-TS and $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$ then the following results are true:

- *(i)* $\mathcal{A}^{Nu-int} \subseteq \mathcal{A}$.
- *(ii)* $\mathcal{X}^{Nu-int} \subseteq \mathcal{X}$, $\emptyset^{Nu-int} = \emptyset$.
- *(iii)* $(\mathcal{A}^{Nu-int})^{Nu-int} = \mathcal{A}^{Nu-int}$.
- *(iv)* *If $\mathcal{A} \subseteq \mathcal{B}$ then* $\mathcal{A}^{Nu-int} \subseteq \mathcal{B}^{Nu-int}$.
- *(v)* $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \subseteq \mathcal{A}^{Nu-int} \cap \mathcal{B}^{Nu-int}$.
- *(vi)* $\mathcal{A}^{Nu-int} \cup \mathcal{B}^{Nu-int} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-int}$.

Proof:

(i) Consider an element x that belongs to the *Nu-interior* of $\mathcal{A}$ then x is contained in some *N-OS* $\mathcal{B}$ which is itself contained in the set $\mathcal{A}$ and hence the result.

(ii) By *(i)*, $\mathcal{X}^{Nu-int} \subseteq \mathcal{X}$.

By *(i)*, $\emptyset^{Nu-int} \subseteq \emptyset$ and $\emptyset \subseteq \emptyset^{Nu-int}$ and so, $\emptyset^{Nu-int} = \emptyset$.

(iii) Let $\mathcal{A}^{Nu-int} = \mathcal{U} = \cup \{\mathcal{U}_i\text{: each } \mathcal{U}_i \text{ is } N\text{-}OS\}$, then $(\mathcal{A}^{Nu-int})^{Nu-int} = (\mathcal{U})^{Nu-int} = \cup \{\mathcal{U}_i\text{: each } \mathcal{U}_i \text{ is } N\text{-}OS\} = \mathcal{U} = \mathcal{A}^{Nu-int}$.

(iv) We have by *(i)* $\mathcal{A}^{Nu-int} \subseteq \mathcal{A} \subseteq \mathcal{B}$ and hence $\mathcal{A}^{Nu-int} \subseteq \mathcal{B}$. Now, $\mathcal{A}^{Nu-\mathrm{int}}$ is a *N-OS* which is contained in $\mathcal{B}$ and so it will either be the *Nu-interior* of $\mathcal{B}$ or contained in the *Nu-interior* of $\mathcal{B}$. That is, $\mathcal{A}^{Nu-int} = \mathcal{B}^{Nu-int}$ or $\mathcal{A}^{Nu-int} \subseteq \mathcal{B}^{Nu-int} \subseteq \mathcal{B}$. In either case, $\mathcal{A}^{Nu-int} \subseteq \mathcal{B}^{Nu-int}$ if $\mathcal{A} \subseteq \mathcal{B}$.

(v) The fact that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ with *(iii)* above will give: $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \subseteq \mathcal{A}^{Nu-int}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ with *(iii)* gives: $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \subset \mathcal{B}^{Nu-int}$ thereby showing that $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \subseteq \mathcal{A}^{Nu-int} \cap \mathcal{B}^{Nu-int}$.

(vi) Let $x \in \mathcal{A}^{Nu-int} \cup \mathcal{B}^{Nu-int} \Rightarrow x \in \mathcal{A}^{Nu-int}$ or $x \in \mathcal{B}^{Nu-int}$

$\Rightarrow x \in \mathcal{Q}_{\mathcal{A}}, \mathcal{Q}_{\mathcal{A}} \subseteq \mathcal{A}$ or, $x \in \mathcal{Q}_{\mathcal{B}} \subseteq \mathcal{B} \Rightarrow x \in \mathcal{Q}_{\mathcal{A}} \cup \mathcal{Q}_{\mathcal{B}} \subseteq \mathcal{A} \cup \mathcal{B}$

$\Rightarrow x \in (\mathcal{A} \cup \mathcal{B})^{Nu-int}$ and hence $\mathcal{A}^{Nu-int} \cup \mathcal{B}^{Nu-int} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-int}$.

Remark 2.1.2

Equality will not hold in the relation $\mathcal{X}^{Nu-int} \subseteq \mathcal{X}$ of (ii) in general. This is because in a *N-TS* $(\mathcal{X}, \mathcal{T})$, the set $\mathcal{X}$ may not be *N-OS* and union of all the subsets of the *N-TS* may also be not equal to $\mathcal{X}$. The following example may be considered. Let $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{3\}, \{1,4\}, \{1,3,4\}\}$. Then it is clear that $(\mathcal{X}, \mathcal{T})$ is a *N-TS* and it can be seen that $\mathcal{X}^{Nu-int} = \{1,3,4\} \neq \mathcal{X}$.

In the case of *GTS* equality holds in case of the result (v) which however is not the case in *N-TS*, and the following example illustrates the fact. Consider $\mathcal{X} = \{1,2,3,4\}$ and let $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}, \{2,3\}, \{3,4\}\}$ then obviously $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. Let $\mathcal{A} = \{3,4\}$ and $\mathcal{B} = \{2,3\}$ then $\mathcal{A} \cap \mathcal{B} = \{3\}$ and $(\mathcal{A} \cap \mathcal{B})^{Nu-int} = \emptyset$. Also, $\mathcal{A}^{Nu-int} = \{3,4\}$ and $\mathcal{B}^{Nu-int} = \{2,3,\}$, so $\mathcal{A}^{Nu-int} \cap \mathcal{B}^{Nu-int} = \{3\}$. Thus, it can be seen that in the case of a *N-TS* $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \neq \mathcal{A}^{Nu-int} \cap \mathcal{B}^{Nu-int}$ in general.

Remark 2.1.3

The reason for (v) of ***proposition 2.1.2*** not holding for the equality sign unlike in the case of *GTS* is because intersection of *N-OS* may not be always *N-O*. Thus, if the intersection of *N-OS* is *N-OS* in a *N-TS* then the properties of interior become analogous to that of *GTS*.

Definition 2.1.2

If $(\mathcal{X},\mathcal{T})$ *is an N-TS, and* $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$*, then the neutro-interior operator on the space* $\mathcal{X}$ *is a function:* $Nu-int$*:* $\mathcal{I}(\mathcal{X}) \rightarrow \mathcal{I}(\mathcal{X})$ *that satisfy:*

(i) $\mathcal{X}^{Nu-int} \subseteq \mathcal{X}$ or, $\emptyset^{Nu-int} = \emptyset$

(ii) $\mathcal{A}^{Nu-int} \subseteq \mathcal{A}$

(iii) $(\mathcal{A}^{Nu-int})^{Nu-int} = \mathcal{A}$

(iv) $(\mathcal{A} \cap \mathcal{B})^{Nu-int} \subseteq \mathcal{A}^{Nu-int} \cap \mathcal{B}^{Nu-int}$

(v) $\mathcal{A}^{Nu-int} \cup \mathcal{B}^{Nu-int} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-int}$

2.2 Exterior in Neutro-Topological Spaces

Definition 2.2.1

Let $(\mathcal{X},\mathcal{T})$ *be a N-TS on the set* $\mathcal{X}$ *and* $\mathcal{A} \subseteq \mathcal{X}$*, then the Neutro-Exterior (Nu-exterior) of* $\mathcal{A}$ *is defined as the union of subsets of* $c\mathcal{A}$ *which are N-OS and is denoted by* $\mathcal{A}^{Nu-ext}$*. That is,* $\mathcal{A}^{Nu-ext} = \cup \{\mathcal{V}_i : \mathcal{V}_i \subseteq c\mathcal{A}$ *and each* $\mathcal{V}_i$ *is N-OS* $\}$*. We define:* $\mathcal{X}^{Nu-ext} = \emptyset$ *and* $\emptyset^{Nu-ext} = \mathcal{X}$*.*

Remark 2.2.1

$\mathcal{A}^{Nu-\text{ext}}$ is the union of all subsets of the *N-T* that do not intersect $\mathcal{A}$. Thus, $\mathcal{A}^{Nu-ext}$ is larger than any other *N-OS* not intersecting $\mathcal{A}$.

Proposition 2.2.1

Let $(\mathcal{X},\mathcal{T})$ *be a N-TS on the set* $\mathcal{X}$ *and* $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$*, then the following are true:*

(i) $\mathcal{A}^{Nu-ext} \subseteq c\mathcal{A}$

(ii) $\mathcal{A}^{Nu-ext} = (c\mathcal{A})^{Nu-int}$

(iii) $\mathcal{A}^{Nu-ext} = [c(\mathcal{A}^{Nu-ext})]^{Nu-ext}$

(iv) $\mathcal{A}^{Nu-int} = (c\mathcal{A})^{Nu-ext}$

(v) *If* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\mathcal{A}^{Nu-ext} \supseteq \mathcal{B}^{Nu-ext}$

(vi) $(\mathcal{A}^{Nu-ext})^{Nu-ext} \supseteq \mathcal{A}^{Nu-int}$

(vii) $(\mathcal{A} \cup \mathcal{B})^{Nu-ext} \subseteq \mathcal{A}^{Nu-ext} \cap \mathcal{B}^{Nu-ext}$

(viii) $\mathcal{A}^{Nu-ext} \cup \mathcal{B}^{Nu-ext} \subseteq (\mathcal{A} \cap \mathcal{B})^{Nu-ext}$

(ix) If $\mathcal{A}$ is a *N-CS* then $\mathcal{A}^{Nu-ext} = c\mathcal{A}$

(x) $\mathcal{A}^{Nu-int} \cap \mathcal{A}^{Nu-ext} = \emptyset$

Proof:

(i) $\mathcal{A}^{Nu-ext} = (c\mathcal{A})^{Nu-int} \subseteq \mathrm{c}\mathcal{A}$ by the definition of Nu-interior.

(ii) By definition: $\mathcal{A}^{Nu-ext} = \cup\, \{\mathcal{V}_{\mathrm{i}}: \mathcal{V}_{\mathrm{i}} \subseteq c\mathcal{A}$ and each $\mathcal{V}_{\mathrm{i}}$ is *N-OS*$\}$

$= (c\mathcal{A})^{Nu-\mathrm{int}}$, by definition of Nu-interior

(iii) We have: $[c(\mathcal{A}^{Nu-ext})]^{Nu-ext}$

$= [c(c\mathcal{A})^{Nu-int}]^{Nu-ext}$, by (ii).

$= [c\{c(c\mathcal{A})^{Nu-int}\}]^{Nu-int}$, by (ii)

$= [(c\mathcal{A})^{Nu-int}]^{Nu-int}$

$= (c\mathcal{A})^{Nu-int}$, [*by* ***proposition 2.1.2 (iii)***]

$= \mathcal{A}^{Nu-ext}$

(iv) We have: $(c\mathcal{A})^{Nu-ext}$

$= \cup\, \{\mathcal{V}_i: \mathcal{V}_i \subseteq c(c\mathcal{A})$ and each $\mathcal{V}_i$ is *N-OS*$\}$

$= \cup\, \{\mathcal{V}_{\mathrm{i}}: \mathcal{V}_{\mathrm{i}} \subseteq \mathcal{A}$ and each $\mathcal{V}_{\mathrm{i}}$ is *N-OS*$\}$

$= \mathcal{A}^{Nu-int}$.

(v) We have: $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c\mathcal{B} \subseteq c\mathcal{A}$

$\Rightarrow (c\mathcal{B})^{Nu-int} \subseteq (c\mathcal{A})^{Nu-int}$ [by ***proposition 2.1.2 (iv)***]

$\Rightarrow \mathcal{B}^{Nu-\mathrm{ext}} \subseteq \mathcal{A}^{Nu-\mathrm{ext}}$

(vi) By *(i)* since $\mathcal{A}^{Nu-ext} \subseteq \mathrm{c}\mathcal{A}$ so by *(v)* we have:

$(\mathcal{A}^{Nu-ext})^{Nu-ext} \supseteq (c\mathcal{A})^{Nu-ext}$

$= (cc\mathcal{A})^{Nu-int}$ [by *(ii)*]

$= \mathcal{A}^{Nu-int}$

Hence $(\mathcal{A}^{Nu-ext})^{Nu-ext} \supseteq \mathcal{A}^{Nu-int}$.

(vii) We have $(\mathcal{A} \cup \mathcal{B})^{Nu-ext}$

$= (c(\mathcal{A} \cup \mathcal{B}))^{Nu-int}$

$= (c\mathcal{A} \cap c\mathcal{B})^{Nu-int}$

$\subseteq (c\mathcal{A})^{Nu-int} \cap (c\mathcal{B})^{Nu-int}$ [by ***proposition 2.1.2 (v)*]**

$= \mathcal{A}^{Nu-ext} \cap \mathcal{A}^{Nu-ext}$

Hence, $(\mathcal{A} \cup \mathcal{B})^{Nu-ext} \subseteq \mathcal{A}^{Nu-ext} \cap \mathcal{B}^{Nu-ext}$.

(viii) We have: $\mathcal{A}^{Nu-ext} \cup \mathcal{B}^{Nu-ext}$

$= (c\mathcal{A})^{Nu-int} \cup (c\mathcal{B})^{Nu-int}$

$\subseteq (c\mathcal{A} \cup c\mathcal{B})^{Nu-int}$ [by ***proposition 2.1.2 (vi)***]

$= (c(\mathcal{A} \cap \mathcal{B}))^{Nu-int}$

$= (\mathcal{A} \cap \mathcal{B})^{Nu-ext}$.

Hence, $\mathcal{A}^{Nu-ext} \cup \mathcal{B}^{Nu-ext} \subseteq (\mathcal{A} \cap \mathcal{B})^{Nu-ext}$

(ix) We have $\mathcal{A}^{Nu-ext} = (c\mathcal{A})^{Nu-int} =$ c$\mathcal{A}$ since, if $\mathcal{A}$ is *N-CS* then c$\mathcal{A}$ is *N-OS* and so $(c\mathcal{A})^{Nu-int} = c\mathcal{A}$. Hence the result.

(x) Let $x \in \mathcal{A}^{Nu-int} \cap \mathcal{A}^{Nu-ext}$

Then $x \in \mathcal{A}^{Nu-int}$ and $x \in \mathcal{A}^{Nu-ext}$

$\Rightarrow x \in \mathcal{A}^{Nu-int} \subseteq \mathcal{A}$ and $x \in (c\mathcal{A})^{Nu-int}$

$\Rightarrow x \in \mathcal{A}^{Nu-int} \subseteq \mathcal{A}$ and $x \in (c\mathcal{A})^{Nu-int} \subseteq c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}$ and $x \in c\mathcal{A}$ which is however not possible as $\mathcal{A} \cap c\mathcal{A} = \emptyset$.

Hence, $\mathcal{A}^{Nu-int} \cap \mathcal{A}^{Nu-ext} = \emptyset$.

Definition 2.2.2

Let $(\mathcal{X}, \mathcal{T})$ be a N-TS, and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the neutro-exterior operator on the space $\mathcal{X}$ is a function: $Nu - ext\colon \mathcal{I}(\mathcal{X}) \to \mathcal{I}(\mathcal{X})$ that satisfy:

(i) $\mathcal{X}^{Nu-ext} = \emptyset$ *or,* $\emptyset^{Nu-ext} = \mathcal{X}$

(ii) $\mathcal{A}^{Nu-ext} \subseteq c\mathcal{A}$

(iii) $\mathcal{A}^{Nu-ext} = (c(\mathcal{A}^{Nu-ext}))^{Nu-ext}$

(iv) $(\mathcal{A} \cup \mathcal{B})^{Nu-ext} \subseteq \mathcal{A}^{Nu-ext} \cap \mathcal{B}^{Nu-ext}$

2.3 Closure in Neutro-Topological Spaces

Proposition 2.3.1

Let $(\mathcal{X}, \mathcal{T})$ be any N-TS, then any one of the following (i), (ii) or (iii) will satisfy in the N-TS:

(i) *Either the null set ($\emptyset$) or the whole set $\mathcal{X}$ is not a N-CS.*

(ii) *Union of some members of $\mathcal{T}$ is not N-CS.*

(iii) *Intersection of some members of $\mathcal{T}$ is not N-CS.*

Definition 2.3.1

Let $(\mathcal{X}, \mathcal{T})$ *be a N-TS with* $\mathcal{A} \subseteq \mathcal{X}$*, then the neutro-closure of* $\mathcal{A}$ *will be the intersection of the N-C supersets of* $\mathcal{A}$ *and will be written in short as Nu-closure and denoted by* $\mathcal{A}^{Nu-cl}$*. Thus,* $\mathcal{A}^{Nu-cl} = \cap \{\mathcal{W}: \mathcal{A} \subseteq \mathcal{W}$ *and* $\mathcal{W}$ *is N-CS*$\}$*. We define:* $\mathcal{X}^{Nu-cl} = \mathcal{X}$ *and* $\emptyset^{Nu-cl} = \emptyset$.

Proposition 2.3.2

If $\mathcal{A}$ *is a N-CS then* $\mathcal{A}^{Nu-cl} = \mathcal{A}$.

Remark 2.3.1

The converse of the above proposition may not be true and can be observed from the example that follows. Let $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{2,3\}, \{3,4\}, \{2,3,4\}\}$. The *N-C* subsets are: $\mathcal{X}, \{2,3,4\}, \{1,4\}, \{1,2\}, \{1\}$.

Consider $\mathcal{A} = \{2\}$, then we have $\mathcal{A}^{Nu-\mathrm{cl}} = \{2,3,4\} \cap \{1,2\} = \{2\} = \mathcal{A}$. But $\mathcal{A}$ itself is not a *N-CS*.

Remark 2.3.2

The Nu-closure of a subset $\mathcal{A}$ of a *N-TS* $(\mathcal{X}, \mathcal{T})$ is not the largest *N-CS* that contain $\mathcal{A}$. This can be seen from ***remark 2.3.1***.

Proposition 2.3.3

Let $(\mathcal{X}, \mathcal{T})$ *be a N-TS and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$*, then the following are true:*

- *(i)* $\mathcal{A} \subseteq \mathcal{A}^{Nu-cl}$
- *(ii)* $(\mathcal{A}^{Nu-cl})^{Nu-cl} = \mathcal{A}^{Nu-cl}$
- *(iii)* $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{Nu-cl} \subseteq \mathcal{B}^{Nu-cl}$
- *(iv)* $\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$
- *(v)* $(\mathcal{A} \cap \mathcal{B})^{Nu-cl} \subseteq \mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl}$

Proof:

- *(i)* By definition 2.3.1.
- *(ii)* We have $\mathcal{A}^{Nu-cl} = \cap \{\mathcal{C}: \mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{W}$ is $N-CS\} = \mathcal{B}$ (say). Here $\mathcal{B}$ is the smallest superset of $\mathcal{A}$. If $\mathcal{B}$ is *N-CS*, then $\mathcal{B}^{Nu-cl} = \mathcal{B}$ by ***proposition 2.3.2*** and we have the result. However, if $\mathcal{B}$ is not *N-CS*, which is possible by ***remarks 2.3.1*** and ***2.3.2***, then $\mathcal{B}^{Nu-cl} = \cap \{\mathcal{E}: \mathcal{B} \subseteq \mathcal{E}$ and $\mathcal{E}$ is $N-CS\} = \cap \{\mathcal{F}: \mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{F}$ is $N-CS\} = \mathcal{B}$, because $\mathcal{B}$ is the smallest superset of $\mathcal{A}$

and there will be no other supersets other than those that are larger than $\mathcal{B}$ and all of which are supersets of $\mathcal{A}$.

(iii) We have by *(i)* $\mathcal{A} \subseteq \mathcal{A}^{Nu-cl}$ and $\mathcal{B} \subseteq \mathcal{B}^{Nu-cl}$.

Now, $\mathcal{B}^{Nu-cl} = \cap \{\mathcal{E}: \mathcal{B} \subseteq \mathcal{E};\ \mathcal{E} \text{ is } N-CS\}$

Now, $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{Nu-cl} = \cap \{\mathcal{F}: \mathcal{A} \subseteq \mathcal{F}, \mathcal{F} \text{ is } N-CS\}$

$\subseteq \cap \{\mathcal{E}: \mathcal{B} \subseteq \mathcal{E}, \mathcal{F} \subseteq \mathcal{E},\ \mathcal{E} \text{ is } N-CS\}$

$= \mathcal{B}^{Nu-cl}$

Hence, $\mathcal{A}^{Nu-cl} \subseteq \mathcal{B}^{Nu-cl}$.

(iv) By *(iii)*, $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{A}^{Nu-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$

Also, $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$

Hence $\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$ and hence the result.

(v) By *(iii)*, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{Nu-cl} \subseteq \mathcal{A}^{Nu-cl}$

Also, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{Nu-cl} \subseteq \mathcal{B}^{Nu-cl}$

Hence $(\mathcal{A} \cap \mathcal{B})^{Nu-cl} \subseteq \mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl}$and hence the result.

Remark 2.3.3

From ***theorem 1.6.3*** we have: $\mathcal{A}^{cl} \cup \mathcal{B}^{cl} = (\mathcal{A} \cup \mathcal{B})^{cl}$, but ***proposition 2.3.3 (iv)*** shows that the equality does not hold in the case of *N-TS* and can be seen from the example below:

Consider $\mathcal{X} = \{1,2,3,4,5\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1,3\}, \{2,5\}, \{1,2,3\}, \{2,3,5\}, \{2,4,5\}\}$ where the *N-CS* are: $\mathcal{X}, \{2,3,4,5\}, \{1,3,4,5\}, \{1,2,3,5\}, \{2,4,5\}, \{1,3,4\}, \{4,5\}, \{1,4\}, \{1,3\}$. Consider $\mathcal{A} = \{1,4\}$ and $\mathcal{B} = \{2,5\}$, then $\mathcal{A}^{Nu-cl} = \{1,4\}$ and $\mathcal{B}^{Nu-cl} = \mathcal{X} \cap \{2,3,4,5\} \cap \{1,2,3,5\} \cap \{2,4,5\} = \{2,5\}$ and as such $\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl} = \{1,4\} \cup \{2,5\} = \{1,2,4,5\}$. Now, $\mathcal{A} \cup \mathcal{B} = \{1,2,4,5\}$ and as such we have $(\mathcal{A} \cup \mathcal{B})^{\mathrm{Nu-cl}} = \mathcal{X}$. Hence, $\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl} \neq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$.

Definition 2.3.2

Let $(\mathcal{X}, \mathcal{T})$ be a N-TS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the neutro-closure operator on the space $\mathcal{X}$ is a function: $Nu-cl$: $\mathcal{I}(\mathcal{X}) \to \mathcal{I}(\mathcal{X})$ that satisfy:

(i) $\mathcal{X}^{Nu-cl} = \mathcal{X}$ or, $\emptyset^{Nu-cl} = \emptyset$

(ii) $\mathcal{A} \subseteq \mathcal{A}^{Nu-cl}$

(iii) $\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Nu-cl}$

(iv) $(\mathcal{A}^{Nu-cl})^{Nu-cl} = \mathcal{A}^{Nu-cl}$

Proposition 2.3.4

If $(\mathcal{X},\mathcal{T})$ *is a N-TS with* $\mathcal{A} \subseteq \mathcal{X}$, *the we have the following relations between the Nu-interior and Nu-closure:*

(i) $c(\mathcal{A}^{Nu-int}) = (c\mathcal{A})^{Nu-cl}$

(ii) $(c\mathcal{A})^{Nu-int} = c(\mathcal{A}^{Nu-cl})$

(iii) $\mathcal{A}^{Nu-int} = c((c\mathcal{A})^{Nu-cl})$

(iv) $c((c\mathcal{A})^{Nu-int}) = \mathcal{A}^{Nu-cl}$.

(v) $(\mathcal{A} \setminus \mathcal{B})^{Nu-int} = \mathcal{A}^{Nu-int} \setminus \mathcal{B}^{Nu-cl}$

(vi) $(\mathcal{A} \setminus \mathcal{B})^{Nu-cl} = \mathcal{A}^{Nu-cl} \setminus \mathcal{B}^{Nu-int}$

Proof:

(i) We have: $\mathcal{A}^{Nu-int} = \cup\, \mathcal{O}_i$ so that each $\mathcal{O}_i$ is *N-OS* and $\mathcal{O}_i \subseteq \mathcal{A}$.

Thus, $c\left(\mathcal{A}^{Nu-int}\right) = c(\cup\, \mathcal{O}_i)$ so that $c(\mathcal{O}_i) \supseteq c(\mathcal{A})$

Or, $c\left(\mathcal{A}^{Nu-int}\right) = \cap\, (c\mathcal{O}_i)$ so that each $c\mathcal{O}_i$ is *N-CS* and $c(\mathcal{A}) \subseteq c(\mathcal{O}_i)$

Or, $c\left(\mathcal{A}^{Nu-int}\right) = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *N-CS* and $c(\mathcal{A}) \subseteq \mathcal{C}_i$

Or, $c\left(\mathcal{A}^{Nu-int}\right) = (c\mathcal{A})^{Nu-cl}$

(ii) We have: $\mathcal{A}^{Nu-cl} = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *N-CS* and $\mathcal{A} \subseteq \mathcal{C}_{\mathrm{i}}$.

Thus, $c(\mathcal{A}^{Nu-cl}) = c(\cap\, \mathcal{C}_i)$ so that $c(\mathcal{A}) \supseteq c(\mathcal{C}_i)$

Or, $c(\mathcal{A}^{Nu-cl}) = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *N-OS* and $c(\mathcal{C}_i) \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{Nu-cl}) = \cup\, (\mathcal{O}_i)$ so that each $\mathcal{O}_i$ is *N-OS* and $\mathcal{O}_i \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{Nu-cl}) = (c\mathcal{A})^{Nu-int}$

(iii) We have $(c\mathcal{A})^{Nu-cl} = \cap\, \mathcal{C}_i$, where each$\mathcal{C}_i$ is *N-CS* and $c\mathcal{A} \subseteq \mathcal{C}_i$.

So, $c(c\mathcal{A})^{Nu-cl} = c(\cap\, \mathcal{C}_i)$ so that $c(c\mathcal{A}) \supseteq c\mathcal{C}_i$

Or, $c(c\mathcal{A})^{Nu-cl} = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *N-OS* and $c\mathcal{C}_i \subseteq \mathcal{A}$

Or, $c(c\mathcal{A})^{Nu-cl} = \cup\, (\mathcal{D}_i)$ so that each $\mathcal{D}_i$ is *N-OS* and $\mathcal{D}_i \subseteq \mathcal{A}$.

Or, $c(c\mathcal{A})^{Nu-cl} = \mathcal{A}^{Nu-int}$

(iv) We have $(c\mathcal{A})^{Nu-int} = \cup\, \mathcal{B}_i$ so that each $\mathcal{B}_i$ is *N-OS* and $\mathcal{B}_i \subseteq c\mathcal{A}$

So, $c((c\mathcal{A})^{Nu-int}) = c(\cup\, \mathcal{B}_i)$ so that each $\mathcal{B}_i$ is *N-OS* and $\mathcal{B}_i \subseteq c\mathcal{A}$

Or, $c((c\mathcal{A})^{Nu-int}) = \cap\, (c\mathcal{B}_i)$ so that each $c\mathcal{B}_i$ is *N-CS* and $c\mathcal{B}_i \supseteq c(c\mathcal{A})$

Or, $c((c\mathcal{A})^{Nu-int}) = \cap\, (\mathcal{C}_i)$ so that each $\mathcal{C}_i$ is *N-CS* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Or, $c((c\mathcal{A})^{Nu-int}) = \mathcal{A}^{Nu-cl}$

(v) We have: $\mathcal{A}^{Nu-int} \setminus \mathcal{B}^{Nu-cl} = \mathcal{A}^{Nu-int} \cap c(\mathcal{B}^{Nu-cl})$

$= \mathcal{A}^{Nu-int} \cap (c\mathcal{B})^{Nu-int}$ [by *(ii)*]

$\supseteq (\mathcal{A} \cap c\mathcal{B})^{Nu-int}$ [by ***proposition 2.1.2 (v)***]

$= (\mathcal{A} \setminus \mathcal{B})^{Nu-int}$

Hence, $(\mathcal{A} \setminus \mathcal{B})^{Nu-int} \subseteq \mathcal{A}^{Nu-int} \setminus \mathcal{B}^{Nu-cl}$

Conversely, let $x \in \mathcal{A}^{Nu-int} \setminus \mathcal{B}^{Nu-cl} \Rightarrow x \in \mathcal{A}^{Nu-int}$ but $x \notin \mathcal{B}^{Nu-cl}$ $\Rightarrow x \in \mathcal{A}$ but $x \notin \mathcal{B} \Rightarrow x \in \mathcal{A} \setminus \mathcal{B} \Rightarrow x \in (\mathcal{A} \setminus \mathcal{B})^{Nu-int}$ as $x \in \mathcal{A}^{Nu-int}$ and so we have $\mathcal{A}^{Nu-int} \setminus \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A} \setminus \mathcal{B})^{Nu-int}$ and hence the result.

(vi) We have: $(\mathcal{A} \setminus \mathcal{B})^{Nu-cl} = (\mathcal{A} \cap c\mathcal{B})^{Nu-cl}$

$\subseteq \mathcal{A}^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}$ [by ***proposition 2.1.2 (v)*****]**

$= \mathcal{A}^{Nu-cl} \cap c(\mathcal{B}^{Nu-int})$ [by *(i)*]

$= \mathcal{A}^{Nu-cl} \setminus \mathcal{B}^{Nu-int}$

Thus, $(\mathcal{A} \setminus \mathcal{B})^{Nu-cl} \subseteq \mathcal{A}^{Nu-cl} \setminus \mathcal{B}^{Nu-int}$.

Conversely, let $x \in \mathcal{A}^{Nu-cl} \setminus \mathcal{B}^{Nu-int} \Rightarrow x \in \mathcal{A}^{Nu-cl}$ but $x \notin \mathcal{B}^{Nu-int}$ $\Rightarrow x \in \mathcal{A}^{Nu-cl}$ but $x \notin \mathcal{B} \Rightarrow x \in (\mathcal{A} \setminus \mathcal{B})^{Nu-cl}$ since $x \in \mathcal{A}^{Nu-cl}$

Thus, $\mathcal{A}^{Nu-cl} \setminus \mathcal{B}^{Nu-int} \subseteq (\mathcal{A} \setminus \mathcal{B})^{Nu-cl}$, and hence the result.

Definition 2.3.4

A N-TS $(\mathcal{X}, \mathcal{T})$ will be called a neutro-door space if and only if every subset of $\mathcal{X}$ is either N-OS or N-CS.

Example 2.3.1

Consider that $\mathcal{X} = \{1,2,3\}$ and $\mathcal{T} = \{\emptyset, \{3\}, \{1,3\}, \{2,3\}\}$ then $(\mathcal{X}, \mathcal{T})$ is a *N-TS* as $\mathcal{X} \notin \mathcal{T}$ and the *N-C* subsets are: $\mathcal{X}, \{1,2\}, \{2\}, \{1\}$ and thus $(\mathcal{X}, \mathcal{T})$ is a neutro-door space.

Proposition 2.3.5

For every door space $(\mathcal{X}, \mathcal{T})$, $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ will be a neutro-door space

Proof:

If $(\mathcal{X}, \mathcal{T})$ is a GTS then $(\mathcal{X}, \mathcal{T} \setminus \mathcal{X})$ will be a N-TS.

Proposition 2.3.6

For every door space $(\mathcal{X}, \mathcal{T})$, $(\mathcal{X}, \mathcal{T} \setminus \emptyset)$ will be a neutro-door space.

Proof:

If $(\mathcal{X},\mathcal{T})$ *is a GTS then* $(\mathcal{X},\mathcal{T}\setminus\emptyset)$ *will be a N-TS*

Definition 2.3.5

If $(\mathcal{X},\mathcal{T})$ *be a N-TS,* $\mathcal{A}\subseteq\mathcal{X}$ *is called neutro-dense in* $\mathcal{X}$ *if* $\mathcal{A}^{Nu-cl}=\mathcal{X}$.

Proposition 2.3.7

If $(\mathcal{X},\mathcal{T})$ *be a N-TS,* $\mathcal{A}\subseteq\mathcal{X}$ *be neutro-dense in* $\mathcal{X}$*, then if* $\mathcal{B}$ *is N-CS and* $\mathcal{A}\subseteq\mathcal{B}$*, then* $\mathcal{B}=\mathcal{X}$.

Proof:

By ***proposition 2.3.3 (iii)***, we have $\mathcal{A}\subseteq\mathcal{B}\Rightarrow\mathcal{A}^{Nu-cl}\subseteq\mathcal{B}^{Nu-cl}\Rightarrow\mathcal{X}\subseteq\mathcal{B}$, since $\mathcal{A}$ is neutro-dense in $\mathcal{X}$, and $\mathcal{B}^{Nu-cl}=\mathcal{B}$, since $\mathcal{B}$ is *N-CS* and as such $\mathcal{B}=\mathcal{X}$.

Proposition 2.3.8

If $(\mathcal{X},\mathcal{T})$ *is a N-TS,* $\mathcal{A}\subseteq\mathcal{X}$ *is neutro-dense in* $\mathcal{X}$*, then* $(c\mathcal{A})^{Nu-int}=\emptyset$.

Proof:

By ***proposition 2.3.4 (ii)***, we have $\left(c\mathcal{A}\right)^{Nu-int}=c\left(\mathcal{A}^{Nu-cl}\right)=c(\mathcal{X})=\emptyset$.

Proposition 2.3.9

If $(\mathcal{X},\mathcal{T})$*is a N-TS,* $\mathcal{A}\subseteq\mathcal{X}$ *is neutro-dense in* $\mathcal{X}$*, then* $\mathcal{B}\cap\mathcal{A}\neq\emptyset$ *for any non-empty N-OS* $\mathcal{B}$.

Proof:

We have $\mathcal{B}\subseteq\mathcal{B}^{Nu-cl}\subseteq\mathcal{X}\Rightarrow\mathcal{B}\subseteq\mathcal{B}^{Nu-cl}\subseteq\mathcal{A}^{Nu-cl}\Rightarrow\mathcal{B}\subseteq\mathcal{A}^{Nu-cl}$ which means that there is a *N-OS* $\mathcal{C}$ so that $\mathcal{B}\subseteq\mathcal{C}\subseteq\mathcal{A}^{Nu-cl}\Rightarrow\mathcal{B}\cap\mathcal{A}\neq\emptyset$.

Definition 2.3.6

Let $(\mathcal{X},\mathcal{T})$ *be a N-TS,* $\mathcal{A}\subseteq\mathcal{X}$ *is called neutro-non-dense if* $(\mathcal{A}^{Nu-cl})^{Nu-int}=\emptyset$.

Example 2.3.2

Consider $\mathcal{X}=\{1,2,3,4,5\}$ with $\mathcal{T}=\{\emptyset,\{2\},\{4\},\{1,2\},\{2,3\},\{3,4\}\}$, when $(\mathcal{X},\mathcal{T})$ becomes a *N-TS* and the *N-CS* are: $\mathcal{X},\{1,3,4,5\},\{1,2,3,5\},\{1,4,5\},\{1,2,5\}$. Assume that $\mathcal{A}=\{1,3\}$, then $\mathcal{A}^{Nu-cl}=\{1,3,4,5\}\cap\{1,2,3,5\}=\{1,3,5\}$ and $(\mathcal{A}^{Nu-cl})^{Nu-int}=\emptyset$. Hence the set $\mathcal{A}=\{1,3\}$ is neutro-non-dense.

2.4 Boundary in Neutro-Topological Spaces

Definition 2.4.1

For a N-TS $(\mathcal{X}, \mathcal{T})$ and a subset $\mathcal{A}$ of $\mathcal{X}$, the neutro-boundary of $\mathcal{A}$, denoted by $\mathcal{A}^{Nu-bd}$, is defined as $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}$. In other words, the neutro-boundary of $\mathcal{A}$ consists of all those points that do not belong to the neutro-interior or the neutro-exterior of $\mathcal{A}$.

Example 2.4.1

Let $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{2,3,4\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS* where the *N-CS* are*:* $\mathcal{X}, \{2,3,4\}, \{3,4\}, \{1,4\}, \{1,2\}, \{1\}$.

Let $\mathcal{A} = \{2,3\}$, then $\mathcal{A}^{Nu-int} = \{2,3\}, c\mathcal{A} = \{1,4\}, (c\mathcal{A})^{Nu-int} = \{1\}$.

So, $\mathcal{A}^{Nu-ext} = \{1\}$. Also, $\mathcal{A}^{Nu-cl} = \{2,3,4\}$ and $(c\mathcal{A})^{Nu-cl} = \{1,4\}$.

Thus, $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} = \{4\}$.

Proposition 2.4.1

If $\mathcal{A}$ is any subset of a N-TS $(\mathcal{X}, \mathcal{T})$ then the following are true:

(i) $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$

(ii) $\mathcal{X} = \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext} \cup \mathcal{A}^{Nu-bd}$

(iii) $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \setminus \mathcal{A}^{Nu-int}$

(iv) $\mathcal{A}^{Nu-int} \cup (c\mathcal{A})^{Nu-int} = c(\mathcal{A}^{Nu-bd})$

(v) $\mathcal{A}^{Nu-int} = \mathcal{A} \setminus \mathcal{A}^{Nu-bd}$

(vi) $\mathcal{A}^{Nu-cl} = \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-bd}$.

Proof:

(i) By definition, if $x \in \mathcal{A}^{Nu-bd}$ then $x \notin \mathcal{A}^{Nu-int}$ and $x \notin \mathcal{A}^{Nu-ext}$

$$\Leftrightarrow x \notin \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext}$$

$$\Leftrightarrow x \in c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$$

Hence, $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$

(ii) From *(i)* we have: $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$ which leads to the results: $\mathcal{A}^{Nu-bd} \cap \mathcal{A}^{Nu-int} = \emptyset$ and $\mathcal{A}^{Nu-bd} \cap \mathcal{A}^{Nu-ext} = \emptyset$ thereby leading to the conclusion that: $\mathcal{X} = \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext} \cup \mathcal{A}^{Nu-bd}$

(iii) From *(i)*, we have: $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$

$$= c(\mathcal{A}^{Nu-int}) \cap c(\mathcal{A}^{Nu-ext}) \quad \ldots\ldots\ldots (1)$$

We have: $\mathcal{A}^{Nu-ext} = (c\mathcal{A})^{Nu-int} = c(\mathcal{A}^{Nu-cl})$ ………….. (2)

[by ***proposition 2.3.4 (ii)***]

By ***proposition 2.2.1 (iv)***:

$\mathcal{A}^{Nu-int} = (c\mathcal{A})^{Nu-ext} = c((c\mathcal{A})^{Nu-cl})$ ……….. (3) [by (2)]

Hence, $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int}) \cap c(\mathcal{A}^{Nu-ext})$ [from (1)]

$= c\{c((c\mathcal{A})^{Nu-cl})\} \cap c\{c(\mathcal{A}^{Nu-cl})\}$ [using (2) and (3)]

$= (c\mathcal{A})^{Nu-cl} \cap \mathcal{A}^{Nu-cl}$, since $c(c\mathcal{A}) = \mathcal{A}$.

Now, $(c\mathcal{A})^{Nu-cl} \cap \mathcal{A}^{Nu-cl} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}$

$= \mathcal{A}^{Nu-cl} \setminus c((c\mathcal{A})^{Nu-cl})$

$= \mathcal{A}^{Nu-cl} \setminus \mathcal{A}^{Nu-int}$ [by ***proposition 2.3.4 (iii)***]

Hence, $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \setminus \mathcal{A}^{Nu-int}$

(iv) We have, $c(\mathcal{A}^{Nu-bd}) = c(\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl})$

$\Rightarrow c(\mathcal{A}^{Nu-bd}) = c(\mathcal{A}^{Nu-cl}) \cup c(c\mathcal{A})^{Nu-cl})$

$\Rightarrow c(\mathcal{A}^{Nu-bd}) = (c\mathcal{A})^{Nu-int} \cup \mathcal{A}^{Nu-int}$ [by ***proposition 2.3.4 (ii) and (iii)***]

Hence, $\mathcal{A}^{Nu-int} \cup (c\mathcal{A})^{Nu-int} = c(\mathcal{A}^{Nu-bd})$.

(v) Let $x \in \mathcal{A}^{Nu-int}$

So, $x \in \mathcal{A}$ but $x \notin c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}$ and $x \in \mathcal{A}^{Nu-cl}$ but $x \notin c\mathcal{A}^{Nu-cl}$

$\Rightarrow x \in \mathcal{A}$ and ($x \in \mathcal{A}^{Nu-cl}$ but $x \notin c\mathcal{A}^{Nu-cl}$)

$\Rightarrow x \in \mathcal{A}$ but ($x \notin \mathcal{A}^{Nu-bd}$)

$\Rightarrow x \in \mathcal{A} \setminus \mathcal{A}^{Nu-bd}$

Hence, $\mathcal{A}^{Nu-int} \subseteq \mathcal{A} \setminus \mathcal{A}^{Nu-bd}$.

Conversely, let $x \in \mathcal{A} \setminus \mathcal{A}^{Nu-bd}$. Then $x \in \mathcal{A}$ and $x \notin \mathcal{A}^{Nu-bd}$, so there exist a *N-OS* $\mathcal{O}_x$ that contain x such that $\mathcal{O}_x \cap c\mathcal{A} = \emptyset$ and $x \in \mathcal{O}_x \subseteq \mathcal{A}$ which therefore shows that $x \in \mathcal{A}^{Nu-int}$. Hence, $\mathcal{A} \setminus \mathcal{A}^{Nu-bd} \subseteq \mathcal{A}^{Nu-int}$. Thus, we have: $\mathcal{A}^{Nu-\mathrm{int}} = \mathcal{A} \setminus \mathcal{A}^{Nu-\mathrm{bd}}$

(vi) We have $\mathcal{A}^{Nu-cl} = \cap \{\mathcal{C}: \mathcal{C}$ is *N-CS* with $\mathcal{A} \subseteq \mathcal{C}\}$

Hence, $c(\mathcal{A}^{Nu-cl}) = c[\cap \{\mathcal{C}: \mathcal{C}$ is *N-CS* with $\mathcal{A} \subseteq \mathcal{C}\}]$

$= \cup \{c\mathcal{C}: c\mathcal{C}$ is *N-OS* with $c\mathcal{C} \subseteq c\mathcal{A}\}$

$= \mathcal{A}^{Nu-ext}$

Hence, $c\{c(\mathcal{A}^{Nu-cl})\} = c(\mathcal{A}^{Nu-ext}) = \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-bd}$, by *(ii)*.

Thus, $\mathcal{A}^{Nu-cl} = \mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-bd}$.

Proposition 2.4.2

If $\mathcal{A}$ and $\mathcal{B}$ are arbitrary subsets of a N-TS $\mathcal{X}$ then:

(i) $\emptyset^{Nu-bd} = \emptyset$

(ii) $\mathcal{A}^{Nu-bd} = (c\mathcal{A})^{Nu-bd}$

Proof*:*

(i) By ***proposition 2.4.1 (i)***, we have: $\mathcal{A}^{Nu-bd} = c(\mathcal{A}^{Nu-int} \cup \mathcal{A}^{Nu-ext})$, wherein replacing $\mathcal{A}$ by $\emptyset$, we have: $\emptyset^{Nu-bd} = c\left(\emptyset^{Nu-int} \cup \emptyset^{Nu-ext}\right) = c(\emptyset \cup \mathcal{X}) = c(\mathcal{X}) = \emptyset$.

(ii) We have: $(c\mathcal{A})^{Nu-bd} = (c\mathcal{A})^{Nu-cl} \cap \{c(c\mathcal{A})\}^{Nu-cl} = (c\mathcal{A})^{Nu-cl} \cap \mathcal{A}^{Nu-cl} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} = \mathcal{A}^{Nu-bd}$.

Remark 2.4.1

In a *N-TS* $\mathcal{X}$, the Nu-boundary of $\mathcal{A} \subseteq \mathcal{X}$ is not necessarily *N-CS* and can be seen from the following counter-example:

We may take $\mathcal{X} = \{1,2,3,4,5\}$ with $\mathcal{T} = \{\emptyset, \{2\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. Let $\mathcal{A} = \{1,3\}$, then $c\mathcal{A} = \{2,4,5\}$ and $\mathcal{A}^{Nu-cl} = \{1,3,4,5\} \cap \{1,2,3,5\} = \{1,3,5\}$ and $(c\mathcal{A})^{Nu-cl} = \mathcal{X}$and so $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} = \{1,3,5\}$ which is not a *N-CS*. However, in a *GTS*, the boundary of a subset of a *GTS* is a closed set which is not true in the case of a subset of a *N-TS*. Because of this, the property that $(\mathcal{A}^{Nu-bd})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd}$ does not hold in the case of *N-TS*.

Proposition 2.4.3

If $\mathcal{A}$ and $\mathcal{B}$ are arbitrary subsets of a N-TS $\mathcal{X}$ then the following are true:

(i) $(\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})) \setminus \mathcal{A}^{Nu-int} \subseteq \mathcal{A}^{Nu-bd}$.

(ii) $(\mathcal{A}^{Nu-int})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd}$

(iii) $(\mathcal{A}^{Nu-cl})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd}$

(iv) $(\mathcal{A} \cap \mathcal{B})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$

(v) $(\mathcal{A} \cup \mathcal{B})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$

Proof:

(i) Let $x \in \mathcal{A}^{Nu-cl} \cap (c\mathcal{A}) \setminus \mathcal{A}^{Nu-int}$

Then, $x \in \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})$ and $x \notin \mathcal{A}^{Nu-int}$

$\Rightarrow x \in \mathcal{A}^{Nu-cl}$ and $x \in (c\mathcal{A})$ and $x \notin \mathcal{A}^{Nu-int}$

$\Rightarrow x \in \mathcal{A}^{Nu-cl}$ and $x \in (c\mathcal{A})$ and $x \notin \mathcal{A}$

$\Rightarrow x \in \mathcal{A}^{Nu-cl}$ and $x \in (c\mathcal{A})^{Nu-cl}$

$\Rightarrow x \in \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}$

$\Rightarrow x \in \mathcal{A}^{Nu-bd}$ and hence $(\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})) \setminus \mathcal{A}^{Nu-int} \subseteq \mathcal{A}^{Nu-bd}$.

(ii) We have $(\mathcal{A}^{Nu-int})^{Nu-bd} = (\mathcal{A}^{Nu-int})^{Nu-cl} \cap [\{c(\mathcal{A}^{Nu-int})\}^{Nu-cl}]$

$= (\mathcal{A}^{Nu-int})^{Nu-cl} \cap [((c\mathcal{A})^{Nu-cl})^{Nu-cl}]$ [by ***proposition 2.3.4 (i)***]

$= (\mathcal{A}^{Nu-int})^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}$ [by ***proposition 2.3.3 (ii)***]

$\subseteq \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}$, since $\mathcal{A}^{Nu-int} \subseteq \mathcal{A}$

$= \mathcal{A}^{Nu-bd}$

Hence, $(\mathcal{A}^{Nu-int})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd}$.

(iii) We have, $(\mathcal{A}^{Nu-cl})^{Nu-bd} = (\mathcal{A}^{Nu-cl})^{Nu-cl} \cap (c(\mathcal{A}^{Nu-cl}))^{Nu-cl}$

$= \mathcal{A}^{Nu-cl} \cap (c(\mathcal{A}^{Nu-cl}))^{Nu-cl}$ [by ***proposition 2.3.3 (ii)***]

Now, $\mathcal{A} \subseteq \mathcal{A}^{Nu-cl} \Rightarrow c(\mathcal{A}^{Nu-cl}) \subseteq c\mathcal{A}$

$\Rightarrow (c(\mathcal{A}^{Nu-cl}))^{Nu-cl} \subseteq (c\mathcal{A})^{Nu-cl}$ [by ***proposition 2.3.3 (iii)***]

Thus, $(\mathcal{A}^{Nu-cl})^{Nu-bd} = \mathcal{A}^{Nu-cl} \cap (c(\mathcal{A}^{Nu-cl}))^{Nu-cl}$

$\subseteq \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} = \mathcal{A}^{Nu-bd}$

Hence $(\mathcal{A}^{Nu-cl})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd}$

(iv) By definition:

$(\mathcal{A} \cap \mathcal{B})^{Nu-bd} = (\mathcal{A} \cap \mathcal{B})^{Nu-cl} \cap (c(\mathcal{A} \cap \mathcal{B}))^{Nu-cl}$

$\subseteq [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl}] \cap [(c\mathcal{A} \cup c\mathcal{B})^{Nu-cl}]$

$\subseteq [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl}] \cap [(c\mathcal{A})^{Nu-cl} \cup (c\mathcal{B})^{Nu-cl}]$

$= [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}] \cup [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}]$

$= [\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} \cap \mathcal{B}^{Nu-cl}] \cup [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}]$

$= [\{\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl}\} \cap \mathcal{B}^{Nu-cl}] \cup [\mathcal{A}^{Nu-cl} \cap \{\mathcal{B}^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}\}]$

$= [(\mathcal{A})^{Nu-bd} \cap \mathcal{B}^{Nu-cl}] \cup [\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-bd}]$

$\subseteq (\mathcal{A})^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$, *since* $(\mathcal{A})^{Nu-bd} \cap \mathcal{B}^{Nu-cl} \subseteq (\mathcal{A})^{Nu-bd}$ and also, $\mathcal{A}^{Nu-cl} \cap \mathcal{B}^{Nu-bd} \subseteq (\mathcal{B})^{Nu-bd}$

Hence, $(\mathcal{A} \cap \mathcal{B})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$

(v) By definition:

$(\mathcal{A} \cup \mathcal{B})^{Nu-bd} = (\mathcal{A} \cup \mathcal{B})^{Nu-cl} \cap (c(\mathcal{A} \cup \mathcal{B}))^{Nu-cl}$

$\subseteq [\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl}] \cap [(c\mathcal{A})^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}]$

$= [(c\mathcal{A})^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl}] \cap [\mathcal{A}^{Nu-cl} \cup \mathcal{B}^{Nu-cl}]$

$= [(c\mathcal{A})^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl} \cap \mathcal{A}^{Nu-cl}] \cup [(c\mathcal{A})^{Nu-cl} \cap (c\mathcal{B})^{Nu-cl} \cap \mathcal{B}^{Nu-cl}]$

$= [\{(c\mathcal{A})^{Nu-cl} \cap \mathcal{A}^{Nu-cl}\} \cap (c\mathcal{B})^{Nu-cl}] \cup [(c\mathcal{A})^{Nu-cl} \cap \{(c\mathcal{B})^{Nu-cl} \cap \mathcal{B}^{Nu-cl}\}]$

$= [\mathcal{A}^{Nu-bd} \cap (c\mathcal{B})^{Nu-cl}] \cup [(c\mathcal{A})^{Nu-cl} \cap \mathcal{B}^{Nu-bd}]$

$\subseteq \mathcal{A}^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$, since $\mathcal{A}^{Nu-bd} \cap (c\mathcal{B})^{Nu-cl} \subseteq \mathcal{A}^{Nu-bd}$ and $(c\mathcal{A})^{Nu-cl} \cap \mathcal{B}^{Nu-bd} \subseteq \mathcal{B}^{Nu-bd}$.

Hence, $(\mathcal{A} \cup \mathcal{B})^{Nu-bd} \subseteq \mathcal{A}^{Nu-bd} \cup \mathcal{B}^{Nu-bd}$.

Remark 2.4.2

That the equality does not hold in ***(i), (ii), (iii)*** of ***proposition 2.4.3***, can be illustrated as follows:

(i) For **(i)** let us consider $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}, \{2,3\}, \{3,4\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. Here the *N-CS* are: $\mathcal{X}, \{2,3,4\}, \{3,4\}, \{1,4\}, \{1,2\}$.

Let $\mathcal{A} = \{1,3\}$, then $c\mathcal{A} = \{2,4\}$.

Then, $\mathcal{A}^{Nu-int} = \{1\}$, $\mathcal{A}^{Nu-cl} = \mathcal{X}$ and $(c\mathcal{A})^{Nu-cl} = \{2,3,4\}$

So, $\mathcal{A}^{Nu-bd} = \mathcal{A}^{Nu-cl} \cap (c\mathcal{A})^{Nu-cl} = \mathcal{X} \cap \{2,3,4\} = \{2,3,4\}$.

Hence, $(\mathcal{A}^{Nu-cl} \cap (c\mathcal{A})) \setminus \mathcal{A}^{Nu-int} = \mathcal{X} \cap \{2,4\} \setminus \{1\} = \{2,4\} \subseteq \mathcal{A}^{Nu-bd}$

(ii) For the inequality in ***proposition 2.4.3 (ii)***, let us consider $\mathcal{X} = \{1,2,3,4,5\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{2,4\}, \{3,5\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. Here the *N-CS* are: $\mathcal{X}, \{2,3,4,5\}, \{1,2,4,5\}, \{1,2,3,4\}, \{2,4,5\}, \{1,3,5\}, \{1,2,4\}$.

Let $\mathcal{A} = \{3,4,5\}$, then $c\mathcal{A} = \{1,2\}$, $\mathcal{A}^{Nu-int} = \{3,5\}$, $\mathcal{A}^{Nu-cl} = \{2,3,4,5\}$, $(c\mathcal{A})^{Nu-cl} = \{1,2,4\}$ and so $\mathcal{A}^{Nu-bd} = \{2,3,4,5\} \cap \{1,2,4\} = \{2,4\}$.

Also, $(\mathcal{A}^{Nu-int})^{Nu-cl} = \{3,5\}^{Nu-cl} = \{3,5\}$

And $(c(\mathcal{A}^{Nu-int}))^{Nu-cl} = \{1,2,4\}^{Nu-cl} = \{1,2,4\}$.

So, $(\mathcal{A}^{Nu-int})^{Nu-bd} = \emptyset$.

Thus, $\emptyset = (\mathcal{A}^{Nu-int})^{Nu-bd} \neq \mathcal{A}^{Nu-bd} = \{2,4\}$ and hence the result.

(iii) For the inequality in ***proposition 2.4.3 (iii)***, let us consider $\mathcal{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}\}$, then $(\mathcal{X}, \mathcal{T})$ is a *N-TS*. Here the *N-CS* are: $\mathcal{X}, \{2,3,4\}, \{1,2,3\}, \{3,4\}, \{1,4\}, \{1,2\}$.

Let $\mathcal{A} = \{1,3\}$, then $c\mathcal{A} = \{2,4\}$.

Now, $\mathcal{A}^{Nu-cl} = \{1,2,3\}$, $(c\mathcal{A})^{Nu-cl} = \{2,3,4\}$, and so $\mathcal{A}^{Nu-bd} = \{2,3\}$.

Again, $(\mathcal{A}^{Nu-cl})^{Nu-cl} = \{1,2,3\}$ and $(c(\mathcal{A}^{Nu-cl}))^{Nt-cl} = (\{4\})^{Nu-cl} = \{4\}$ and so, $(\mathcal{A}^{Nu-cl})^{Nu-bd} = (\mathcal{A}^{Nu-cl})^{Nu-cl} \cap (c(\mathcal{A}^{Nu-cl}))^{Nu-cl} = \{1,2,3\} \cap \{4\} = \emptyset$. Thus, we have: $\emptyset = (\mathcal{A}^{Nu-cl})^{Nu-bd} \neq \mathcal{A}^{Nu-bd} = \{2,3\}$.

2.5 Relative Topology of a Neutro-Topological Space

Definition 2.5.1

For a N-TS $(\mathcal{X}, \mathcal{T})$ and $\mathcal{A} \subseteq \mathcal{X}$, we define the neutro-relative topology $\mathcal{T}_{\mathcal{A}}$ for $\mathcal{A}$ to be the collection given by: $\mathcal{T}_{\mathcal{A}} = \{\mathcal{B} \cap \mathcal{A} : \mathcal{B} \in \mathcal{T}\}$. The N-TS $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ is called a sub-space of the N-TS $(\mathcal{X}, \mathcal{T})$ and the N-T $\mathcal{T}_{\mathcal{A}}$ is said to be induced by $\mathcal{T}$.

Proposition 2.5.1

Suppose $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ is a sub-space of a N-TS $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$, then the following results are true:

(i) *$\mathcal{A} \subseteq \mathcal{Y}$ is N-C in $\mathcal{Y}$ iff there is a N-CS $\mathcal{C}$ in $\mathcal{X}$ such that $\mathcal{A} = \mathcal{C} \cap \mathcal{Y}$.*

(ii) *For every $\mathcal{A} \subseteq \mathcal{Y}$, $\mathcal{A}_{\mathcal{Y}}^{Nu-cl} = \mathcal{A}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{Y}$, where $\mathcal{A}_{\mathcal{X}}^{Nu-cl}$ is the Nu-closure of $\mathcal{A}$ in $\mathcal{X}$.*

(iii) *A subset $\mathcal{P}$ of $\mathcal{Y}$ will be a $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd of a point $y \in \mathcal{Y}$ iff $\mathcal{P} = \mathcal{Q} \cap \mathcal{Y}$ for some $\mathcal{T}_{\mathcal{X}}$-neutro-nhd $\mathcal{Q}$ of y.*

(iv) *For every $\mathcal{A} \subseteq \mathcal{Y}$, $\mathcal{A}_{\mathcal{X}}^{Nu-int} \subseteq \mathcal{A}_{\mathcal{Y}}^{Nu-int}$*

Proof:

(i) Let $\mathcal{A}$ be *N-C* in $\mathcal{Y}$

$\Leftrightarrow c\mathcal{A}$ is *N-O* in $\mathcal{Y}$

$\Leftrightarrow c\mathcal{A} = \mathcal{B} \cap \mathcal{Y}$, $\mathcal{B}$ is *N-O* in $\mathcal{X}$

$\Leftrightarrow \mathcal{A} = \mathrm{c}(\mathcal{B} \cap \mathcal{Y})$

$\Leftrightarrow \mathcal{A} = \mathrm{c}(\mathcal{B}) \cup \mathrm{c}(\mathcal{Y})$, De-Morgan's law

$\Leftrightarrow \mathcal{A} = \mathrm{c}(\mathcal{B}) \cup \emptyset$, since $\mathrm{c}(\mathcal{Y}) = \emptyset$

$\Leftrightarrow \mathcal{A} = \mathrm{c}(\mathcal{B}) = \mathcal{Y} \setminus \mathcal{B}$

$\Leftrightarrow \mathcal{A} = \mathcal{Y} \cap \mathrm{c}\mathcal{B}$

$\Leftrightarrow \mathcal{A} = \mathcal{Y} \cap \mathcal{C}$, where $\mathcal{C} = c\mathcal{B}$ is *N-C* in $\mathcal{X}$.

(ii) By definition:

$\mathcal{A}_{\mathcal{Y}}^{Nu-cl} = \cap \{\mathcal{D}: \mathcal{D}$ is *N-CS* in $\mathcal{Y}$ and $\mathcal{A} \subseteq \mathcal{D}\}$

$= \cap \{\mathcal{C} \cap \mathcal{Y}: \mathcal{C}$ is *N-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C} \cap \mathcal{Y}\}$, by *(i)*

$= \cap \{\mathcal{C} \cap \mathcal{Y}: \mathcal{C}$ is *N-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C}\}$

$= [\cap \{\mathcal{C}: \mathcal{C}$ is *N-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C}\}] \cap \mathcal{Y}$

$= \mathcal{A}_{\mathcal{X}}^{Nu-cl} \cap \mathcal{Y}$, where $\mathcal{A}_{\mathcal{X}}^{Nu-cl}$ is the Nu-closure of $\mathcal{A}$ in $\mathcal{X}$.

(iii) Let us assume $\mathcal{P}$ to be a $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd of a point y in $\mathcal{Y}$. Then a $\mathcal{T}_{\mathcal{Y}}$-*N-OS* set $\mathcal{K}$ will be there so that $y \in \mathcal{K} \subseteq \mathcal{P}$.

Thus, for a $\mathcal{T}_{\mathcal{X}}$-*N-OS* $\mathcal{I}$ we have: $y \in \mathcal{K} = \mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{P}$. Now, if we assume $\mathcal{Q} = \mathcal{P} \cup \mathcal{J}$, then $\mathcal{Q}$ is a $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd of y since $\mathcal{I}$ is a $\mathcal{T}_{\mathcal{X}}$-*N-OS* such that $y \in \mathcal{J} \subseteq \mathcal{Q}$.

Further, $\mathcal{Q} \cap \mathcal{Y} = (\mathcal{P} \cup \mathcal{J}) \cap \mathcal{Y} = (\mathcal{P} \cap \mathcal{Y}) \cup (\mathcal{J} \cap \mathcal{Y}) = \mathcal{P} \cup (\mathcal{J} \cap \mathcal{Y}) = \mathcal{P}$, since $\mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{P}$.

Conversely, if $\mathcal{P} = \mathcal{Q} \cap \mathcal{Y}$ for some $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd $\mathcal{Q}$ of y. Then there exists a $\mathcal{J} \in \mathcal{T}_{\mathcal{X}}$ so that $y \in \mathcal{J} \subseteq \mathcal{Q}$ which means $y \in \mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{Q} \cap \mathcal{Y} = \mathcal{P}$. And since $\mathcal{J} \cap \mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}$, so $\mathcal{P}$ is a $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd of the point y.

(iv) We have $x \in \mathcal{A}_{\mathcal{X}}^{Nu-int} \Rightarrow x$ is a $\mathcal{T}_{\mathcal{X}}$-interior of $\mathcal{A} \Rightarrow \mathcal{A}$ is a $\mathcal{T}_{\mathcal{X}}$-neutro-nhd of $x \Rightarrow \mathcal{A} \cap \mathcal{Y}$ is a $\mathcal{T}_{\mathcal{Y}}$-neutro-nhd of $x \Rightarrow \mathcal{A} \subseteq \mathcal{Y} \Rightarrow x \in \mathcal{A}_{\mathcal{Y}}^{Nu-int}$ and hence we must have $\mathcal{A}_{\mathcal{X}}^{Nu-int} \subseteq \mathcal{A}_{\mathcal{Y}}^{Nu-int}$