CHAPTER 3

Neutro-Bi-topological Spaces

In this chapter the notion of neutro-bi-topological space (N-B-TS) is defined and the aspects of interior, exterior, closure, and boundary are defined in neutro-bi-topological spaces (N-B-TS) and the various properties of these aspects that are generally true for GTS are inspected. Further, some new concepts of pseudo-exterior, quasi-open, quasi-closed sets, neutro-quasi-interior, neutro-quasi-closure are defined and some of their properties are analyzed with respect to the N-B-TS.

3.1 Introduction to Neutro-Bi-topological Space

Definition 3.1.1

Let $\mathcal{T}_1, \mathcal{T}_2$ *be two N-Ts on a universe* $\mathcal{X}$, [$\mathcal{T}_1$ *and* $\mathcal{T}_2$ *maybe same or different*] *then the triplet* $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *becomes a neutro-bi-topological space (N-B-TS).*

Remark 3.1.1

The subsets of $\mathcal{X}$ that are included in the *N-T* $\mathcal{T}_1$ will be called *N-O* with respect to $\mathcal{T}_1$ and written as $\mathcal{T}_1$-*N-OS* and those in the *N-T* $\mathcal{T}_2$ will be called *N-O* with respect to $\mathcal{T}_2$ and will be written as $\mathcal{T}_2$-*N-OS*. In this chapter no union or intersection of the members of $\mathcal{T}_1$ and $\mathcal{T}_2$ will be considered and the subsets corresponding to the *N-Ts* will be studied separately. The *N-CSs*, which are the complements of the *N-OSs* will be identified separately with respect to the two *N-Ts* and written as $\mathcal{T}_1$-*N-CS* or $\mathcal{T}_2$-*N-CS.*

Proposition 3.1.1

For every N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{X}, \mathcal{T}_1 \cap \mathcal{T}_2)$ *is a N-TS.*

Proof*:* If $\mathcal{T}_1, \mathcal{T}_2$ are *N-Ts*, then either $\emptyset \in \mathcal{T}_1$ or, $\mathcal{X} \in \mathcal{T}_1$ and either $\emptyset \in \mathcal{T}_2$ or, $\mathcal{X} \in \mathcal{T}_2$. In either case $\emptyset \in \mathcal{T}_1 \cap \mathcal{T}_2$ or, $\mathcal{X} \in \mathcal{T}_1 \cap \mathcal{T}_2$ and it satisfies the first condition for a *N-T.*

Some of the results discussed in this chapter are published in: Basumatary, B., Khaklary, J.K., & Said, B. (2022). On Neutro-Bitopological Spaces. *International Journal of Neutrosophic Science, 18(2), 254-261.*

Remark 3.1.2

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ is a *N-B-TS* then $(\mathcal{X}, \mathcal{T}_1 \cup \mathcal{T}_2)$ may not be a *N-TS*. The reason for this being that if the empty set belong to one of the *N-T* and the whole universe belong to the other *NT*, then the union of the two *N-Ts* will include both the empty set and the whole set and this will not satisfy the first condition for a *N-T*.

Proposition 3.1.2

For every B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset, \mathcal{T}_2 \setminus \emptyset)$ *is a N-B-TS.*

Proof*:* By ***theorem 1.6.15***, it can be seen that if $(\mathcal{X}, \mathcal{T}_1)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset)$ is a *N-T* on $\mathcal{X}$ and similarly, if $(\mathcal{X}, \mathcal{T}_2)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_2 \setminus \emptyset)$ is a *N-T* on $\mathcal{X}$. Thus, since $\mathcal{T}_1 \setminus \emptyset$ and $\mathcal{T}_2 \setminus \emptyset$ are two *N-Ts* on $\mathcal{X}$, so by definition $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset, \mathcal{T}_2 \setminus \emptyset)$ will be a *N-B-TS*.

Proposition 3.1.3

For every B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X}, \mathcal{T}_2 \setminus \mathcal{X})$ *is a N-B-TS.*

Proof*:* By ***theorem 1.6.16***, if $(\mathcal{X}, \mathcal{T}_1)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X})$ is a *N-T* on $\mathcal{X}$ and similarly, if $(\mathcal{X}, \mathcal{T}_2)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_2 \setminus \mathcal{X})$ is a *N-T* on $\mathcal{X}$. Thus, since $\mathcal{T}_1 \setminus \mathcal{X}$ and $\mathcal{T}_2 \setminus \mathcal{X}$ are two *N-Ts* on $\mathcal{X}$, so $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X}, \mathcal{T}_2 \setminus \mathcal{X})$ will be a *N-B-TS.*

Proposition 3.1.4

For every BTS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset, \mathcal{T}_2 \setminus \mathcal{X})$ *is a N-B-TS.*

Proof*:* By ***theorem 1.6.15,*** if $(\mathcal{X}, \mathcal{T}_1)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset)$ is a *N-T* on $\mathcal{X}$ and by ***theorem 1.6.16,*** if $(\mathcal{X}, \mathcal{T}_2)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_2 \setminus \mathcal{X})$ is a *N-T* on $\mathcal{X}$. Thus, since $\mathcal{T}_1 \setminus \emptyset$ and $\mathcal{T}_2 \setminus \mathcal{X}$ are two *N-Ts* on $\mathcal{X}$, so $(\mathcal{X}, \mathcal{T}_1 \setminus \emptyset, \mathcal{T}_2 \setminus \mathcal{X})$ will be a *N-B-TS.*

Proposition 3.1.5

For every B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X}, \mathcal{T}_2 \setminus \emptyset)$ *is a N-B-TS.*

Proof*:* By ***theorem 1.6.16,*** if $(\mathcal{X}, \mathcal{T}_1)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X})$ is a *N-T* on $\mathcal{X}$ and ***theorem 1.6.15***, if $(\mathcal{X}, \mathcal{T}_2)$ is a classical topology on $\mathcal{X}$ then $(\mathcal{X}, \mathcal{T}_2 \setminus \emptyset)$ is a *N-T* on $\mathcal{X}$. Thus, since $\mathcal{T}_1 \setminus \mathcal{X}$ and $\mathcal{T}_2 \setminus \emptyset$ are two *N-Ts* on $\mathcal{X}$, so $(\mathcal{X}, \mathcal{T}_1 \setminus \mathcal{X}, \mathcal{T}_2 \setminus \emptyset)$ will be a *N-B-TS*.

3.2 Interior in Neutro-Bi-topological Spaces

Definition 3.2.1

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *be a N-B-TS and* $\mathcal{A} \subseteq \mathcal{X}$ *then the Nu-Bi-interior of* $\mathcal{A}$ *is denoted by* $\mathcal{A}^{\mathcal{T}_{12}-Nint}$ *and is defined as the Nu-interior with respect to* $\mathcal{T}_1$ *of the Nu-interior of* $\mathcal{A}$ *with respect to* $\mathcal{T}_2$*. That is:* $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$*, where* $\mathcal{A}^{\mathcal{T}_2-Nint} = \cup \{\mathcal{O}_i$: *each* $\mathcal{O}_i$ *is* $\mathcal{T}_2$*-N-OS and* $\mathcal{O}_i \subseteq \mathcal{A}\}$*. Thus,* $(\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint} = \cup \{\mathcal{O}_i$: *each* $\mathcal{O}_i$ *is* $\mathcal{T}_1$*-N-OS and* $\mathcal{O}_i \subseteq \mathcal{A}^{\mathcal{T}_2-Nint}\}$.

Remark 3.2.1

The term Bi-interior has been used because we have taken the interior of the set two times with respect to the two *N-Ts* successively.

Definition 3.2.2

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *be a N-B-TS and* $\mathcal{A} \subseteq \mathcal{X}$*. If* $\mathcal{A}$ *is a N-OS with respect to both* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *then we call such subsets as a* $\mathcal{T}_{12}$*-N-OS.*

Proposition 3.2.1

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *is a N-B-TS and* $\mathcal{A} \subseteq \mathcal{X}$*, and* $\mathcal{A}$ *is* $\mathcal{T}_{12}$*-NOS then* $\mathcal{A}^{\mathcal{T}_{12}-Nint} = \mathcal{A}$.

Proof: By definition: $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$= (\mathcal{A})^{\mathcal{T}_1-Nint}$, since $\mathcal{A}$ is $\mathcal{T}_2$-*N-OS* [by ***proposition 2.1.1***]

$= \mathcal{A}$, since $\mathcal{A}$ is $\mathcal{T}_1$-*N-OS* [by ***proposition 2.1.1***]

Remark 3.2.2

The converse of ***proposition 3.2.1*** is not always true. That is, if $\mathcal{A}^{\mathcal{T}_{12}-int} = \mathcal{A}$ then $\mathcal{A}$ may not be a $\mathcal{T}_{12}$-*N-OS*.

Consider $\mathcal{X} = \{1,2,3,4,5\}$ and let $\mathcal{T}_1 = \{\emptyset, \{1\}, \{3\}, \{2,3\}, \{1,3,4\}$, $\mathcal{T}_2 = \{\emptyset, \{3\}, \{4\}, \{1,2\}, \{2,4\}, \{2,3,4\}\}$ and $\mathcal{A} = \{1,2,3\}$.

Then $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint} = (\{1,2,3\})^{\mathcal{T}_1-Nint} = \{1,2,3\} = \mathcal{A}$, but $\mathcal{A}$ is not $\mathcal{T}_{12}$-*NOS*.

Remark 3.2.3

The counter example provided in ***remark 3.2.2*** also shows that the Nu-Bi-interior of $\mathcal{A}$ is not the biggest *N-OS* contained in $\mathcal{A}$.

Proposition 3.2.2

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$*, if* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$*, then the following results are true.*

(i) $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}$

(ii) $\emptyset^{\mathcal{T}_{12}-Nint} = \emptyset$; $\mathcal{X}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{X}$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{B}^{\mathcal{T}_{12}-Nint}$

(iv) $(\mathcal{A}^{\mathcal{T}_{12}-Nint})^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nint}$

(v) $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nint} \cap \mathcal{B}^{\mathcal{T}_{12}-Nint}$

(vi) $\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right) \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Nint}$

Proof*:*

(i) By definition the result is obvious.

(ii) Since the null set is a subset of all sets, we have $\emptyset \subseteq \emptyset^{\mathcal{T}_{12}-Nint}$ and by *(i)* we have: $\emptyset^{\mathcal{T}_{12}-Nint} \subseteq \emptyset$ and thus, $\emptyset^{\mathcal{T}_{12}-Nint} = \emptyset$.

By *(i)*, $\mathcal{X}^{\mathcal{T}_{12}-\mathrm{Nint}} \subseteq \mathcal{X}$.

(iii) We have: $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$. Now, $\mathcal{A}^{\mathcal{T}_2-Nint} \subseteq \mathcal{B}^{\mathcal{T}_2-Nint}$ since $\mathcal{A} \subseteq \mathrm{B}$ by ***proposition 2.1.2 (iv)***. Again since $\mathcal{A}^{\mathcal{T}_2-\mathrm{Nint}} \subseteq \mathcal{B}^{\mathcal{T}_2-\mathrm{Nint}}$, by ***proposition 2.1.2 (iv)***: $(\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint} \subseteq (\mathcal{B}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$ from which, we get: $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{B}^{\mathcal{T}_{12}-Nint}$

(iv) We have: $\mathcal{A}^{\mathcal{T}_{12}-Nint} = \cup \{\mathcal{O}_i$: each $\mathcal{O}_i$ is $\mathcal{T}_1$*-NOS* and $\mathcal{O}_i \subseteq \mathcal{A}^{\mathcal{T}_2-Nint}\} = \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$ and by *(iii)*, we have: $\mathcal{B}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nint}$ which gives:

$$(\mathcal{A}^{\mathcal{T}_{12}-Nint})^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nint}$$

(v) We have: $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Nint} = ((\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

Now, $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Nint} \subseteq (\mathcal{A}^{\mathcal{T}_2-Nint}) \cap (\mathcal{B}^{\mathcal{T}_2-Nint})$, by ***proposition 2.1.2 (v)*** and so by ***proposition 2.1.2 (iv)***, we have:

$[(\mathcal{A} \cap \mathcal{B})^{[\mathcal{T}_2-Nint}]^{\mathcal{T}_1-Nint} \subseteq [(\mathcal{A}^{\mathcal{T}_2-Nint}) \cap (\mathcal{B}^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint}$ and again, by ***proposition 2.1.2 (v)***, applied on the right side, we get:

$[(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Nint}]^{\mathcal{T}_1-Nint} \subseteq [(\mathcal{A}^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint} \cap [(\mathcal{B}^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint}$

Or, $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Nint} \subseteq \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cap \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right)$

(vi) Let $x \in \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right)$

$\Rightarrow x \in \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right)$ or, $x \in \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right)$. This shows that there exists *N-OSs* $\mathcal{O}_i$ and $\mathcal{O}_j$ such that: $x \in \{\cup\, \mathcal{O}_i\} \subseteq \mathcal{A}^{\mathcal{T}_2-Nint}$ or, $x \in \{\cup\, \mathcal{O}_j\} \subseteq \mathcal{B}^{\mathcal{T}_2-Nint}$

$\Rightarrow x \in \mathcal{O}_i \cup \mathcal{O}_j \subseteq \mathcal{A}^{\mathcal{T}_2-Nint} \cup \mathcal{B}^{\mathcal{T}_2-Nint} \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_2-Nint}$, by ***proposition 2.1.2 (vi)***. And similarly, $x \in (\mathcal{A}^{\mathcal{T}_1-Nint}) \cup (\mathcal{B}^{\mathcal{T}_1-Nint}) \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_1-Nint}$
So $x \in (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Nint} \Rightarrow \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right) \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Nint}$

Remark 3.2.4

Equality will not hold in the case of ***proposition 3.2.2 (iv)*** and can be seen from the following example: Assume $\mathcal{X} = \{1,2,3,4,5\}$, $\mathcal{T}_1 = \{\emptyset, \{1\}, \{3\}, \{2,3\}, \{3,4\}, \{4,5\}\}$, $\mathcal{T}_2 = \{\emptyset, \{2\}, \{3\}, \{1,2\}, \{3,4\}, \{3,4,5\}\}$.
Let $\mathcal{A} = \{1,2\}$, then $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint} = (\{1,2\})^{\mathcal{T}_1-Nint} = \{1\}$ and $(\mathcal{A}^{\mathcal{T}_{12}-Nint})^{\mathcal{T}_{12}-Nint} = (\{1\})^{\mathcal{T}_{12}-Nint} = ((\{1\})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint} = \emptyset^{\mathcal{T}_1-Nint} = \emptyset$.
Thus, we have: $(\mathcal{A}^{\mathcal{T}_{12}-int})^{\mathcal{T}_{12}-Nint} \neq \mathcal{A}^{\mathcal{T}_{12}-Nint}$.
Equality will not hold in ***(v)***. Let us assume that $\mathcal{X} = \{1,2,3,4,5\}$, $\mathcal{T}_1 = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}, 1,2,3\}, \{3,4,5\}$, $\mathcal{T}_2 = \{\emptyset, \{2\}, \{4\}, \{1,3\}, \{2,5\}, \{2,3,4\}, \{1,3,5\}\}$, $\mathcal{A} = \{1,2,3\}$, and $\mathcal{B} = \{2,3,4\}$, then $\mathcal{A} \cap \mathcal{B} = \{2,3\}$ and $\mathcal{A}^{\mathcal{T}_{12}-Nint} = \{1,2,3\}$, $\mathcal{B}^{\mathcal{T}_{12}-Nint} = \{2,3\}$ and $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Nint} = \emptyset$, but $\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cap \left(\mathcal{B}^{\mathcal{T}_{12}-Nint}\right) = \{2,3\}$.
Equality will not hold in ***(vi)*** and can be seen if we consider $\mathcal{A} = \{1,2\}$ and $\mathcal{B} = \{3,4\}$ in the above example.
We then get: $\mathcal{A}^{\mathcal{T}_{12}-Nint} = \emptyset$, $\mathcal{B}^{\mathcal{T}_{12}-Nint} = \emptyset$ and $(\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Nint} = \{1,2,3\}$.

3.3 Closure in Neutro-Bi-topological Spaces

Definition 3.3.1

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ is a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$, then $\mathcal{A}$ will be called $\mathcal{T}_{12}$ -N-CS if the complement of $\mathcal{A}$, i.e. $c\mathcal{A}$ is $\mathcal{T}_{12}$-N-OS.

Definition 3.3.2

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$ then the Nu-Bi-closure of $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathcal{T}_{12}-Ncl}$ and is defined as the Nu-closure with respect to $\mathcal{T}_1$ of the Nu-closure of $\mathcal{A}$ with respect to $\mathcal{T}_2$. That is: $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$, where $\mathcal{A}^{\mathcal{T}_2-Ncl} = \cap \{\mathcal{C}_i$: each $\mathcal{C}_i$ is $\mathcal{T}_2$-N-CS and $\mathcal{A} \subseteq \mathcal{C}_i\}$. Thus, $(\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} = \cap \{\mathcal{C}_i$: each $\mathcal{C}_i$ is $\mathcal{T}_1$-N-CS and $\mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{C}_i\}$. We define: $\emptyset^{\mathcal{T}_{12}-Ncl} = \emptyset$.

Proposition 3.3.1

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A} \subseteq \mathcal{X}$ and if $\mathcal{A}$ is $\mathcal{T}_{12}$-N-CS then $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = \mathcal{A}$.

Proof: If $\mathcal{A}$ is $\mathcal{T}_{12}$-*NCS*, then by ***proposition 2.3.2***, we have:

$\mathcal{A}^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} = (\mathcal{A})^{\mathcal{T}_1-Ncl} = \mathcal{A}$.

Remark 3.3.1

The converse of ***proposition 3.3.1*** is not true. That is, if $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = \mathcal{A}$, then it is not necessary that $\mathcal{A}$ is $\mathcal{T}_{12}$-*N*-*CS*. The following example can be taken to illustrate it. Assume $\mathcal{X} = \{1,2,3,4\}$, $\mathcal{T}_1 = \{\emptyset, \{1\}, \{2\}, \{3,4\}, \{1,3,4\}\}, \mathcal{T}_2 = \{\emptyset, \{3\}, \{4\}, \{1,2\}, \{2,4\}\}$ and $\mathcal{A} = \{1,2\}$, then $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = \{1,2\} = \mathcal{A}$. But $\mathcal{A}$ is not *N*-*CS* with respect to $\mathcal{T}_2$ and so is not $\mathcal{T}_{12}$-*N*-*CS*.

Proposition 3.3.2

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *is a N-BTS and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, *then the results below are true:*

- *(i)* $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$
- *(ii)* $\mathcal{X}^{\mathcal{T}_{12}-Ncl} = \mathcal{X}$
- *(iii)* $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{\mathcal{T}_{12}-Ncl} \subseteq \mathcal{B}^{\mathcal{T}_{12}-Ncl}$
- *(iv)* $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Ncl} \subseteq (\mathcal{A}^{\mathcal{T}_{12}-Ncl}) \cap (\mathcal{B}^{\mathcal{T}_{12}-Ncl})$
- *(v)* $(\mathcal{A}^{\mathcal{T}_{12}-Ncl}) \cup (\mathcal{B}^{\mathcal{T}_{12}-Ncl}) \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Ncl}$

Proof:

(i) We have: $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$

Now, $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_2-Ncl}$, [by ***proposition 2.3.3 (i)***]

So, $\mathcal{A} \subseteq (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} = \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

(ii) By *(i)*, $\mathcal{X} \subseteq \mathcal{X}^{\mathcal{T}_{12}-Ncl}$ and since $\mathcal{X}$ is the universal set, we have: $\mathcal{X}^{\mathcal{T}_{12}-Ncl} \subseteq \mathcal{X}$ and thus $\mathcal{X}^{\mathcal{T}_{12}-Ncl} = \mathcal{X}$.

(iii) Since $\mathcal{A} \subseteq \mathcal{B}$, by ***proposition 2.3.3 (iii)*** we have:

$\mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{B}^{\mathcal{T}_2-Ncl}$ and by the same proposition we again have:

$(\mathcal{A}^{\mathcal{T}_2-cl})^{\mathcal{T}_1-Ncl} \subseteq (\mathcal{B}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$

Thus, $\mathcal{A}^{\mathcal{T}_{12}-cl} \subseteq \mathcal{B}^{\mathcal{T}_{12}-cl}$.

(iv) We have: $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Ncl} = ((\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$

Now, $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Ncl} \subseteq (\mathcal{A}^{\mathcal{T}_2-Ncl}) \cap (\mathcal{B}^{\mathcal{T}_2-Ncl})$ [by ***proposition 2.3.3 (v)***]

Thus, $[(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_2-Ncl}]^{\mathcal{T}_1-Ncl}$

$\subseteq [(\mathcal{A}^{\mathcal{T}_2-Ncl}) \cap (\mathcal{B}^{\mathcal{T}_2-Ncl})]^{\mathcal{T}_1-Ncl}$, [by ***proposition 2.3.3 (iii)***]

$\subseteq (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} \cap (\mathcal{B}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$, [by ***proposition 2.3.3 (v)***]

Thus, $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Ncl} \subseteq (\mathcal{A}^{\mathcal{T}_{12}-cl}) \cap (\mathcal{B}^{\mathcal{T}_{12}-Ncl})$

(v) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B}$, so by *(iii)*, we have:

$\mathcal{A}^{\mathcal{T}_{12}-Ncl} \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Ncl}$ *and* $\mathcal{B}^{\mathcal{T}_{12}-Ncl} \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Ncl}$.

Thus, we have: $(\mathcal{A}^{\mathcal{T}_{12}-Ncl}) \cup (\mathcal{B}^{\mathcal{T}_{12}-Ncl}) \subseteq (\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}-Ncl}$.

Remark 3.3.2

Equality will not always hold in ***proposition 3.3.2 (iv)*** and the following example shows it. Suppose that $\mathcal{X} = \{1,2,3,4,5\}$, $\mathcal{T}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$ and $\mathcal{T}_2 = \{\emptyset, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,5\}, \{2,5\}, \{3,4\}, \{4,5\}\}$. Suppose $\mathcal{A} = \{1,3,4\}$ and $\mathcal{B} = \{4,5\}$, then it can be seen that: $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = \{1,2,4,5\}$ and $\mathcal{B}^{\mathcal{T}_{12}-\mathrm{Ncl}} = \{2,4,5\}$ and as such, we have: $\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap \mathcal{B}^{\mathcal{T}_{12}-Ncl} = \{2,4,5\}$

Now, $\mathcal{A} \cap \mathcal{B} = \{4\}$ and $(\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}-Ncl} = \{4,5\}$

Proposition 3.3.3

For a NBTS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the results that follow are true:

(i) $c(\mathcal{A}^{\mathcal{T}_{12}-Nint}) = (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

(ii) $c(\mathcal{A}^{\mathcal{T}_{12}-Ncl}) = (c\mathcal{A})^{\mathcal{T}_{12}-Nint}$

(iii) $(\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_{12}-Nint} = \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \setminus (\mathcal{A}^{\mathcal{T}_{12}-Ncl})$

(iv) $(\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_{12}-Ncl}) \setminus \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right)$

Proof*:*

(i) By definition we have:

$\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$= c[\{c(\mathcal{A}^{\mathcal{T}_2-Nint})\}^{\mathcal{T}_1-Ncl}]$, [by ***proposition 2.3.4 (iii)***]

$= c[((c\mathcal{A})^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}]$, [by ***proposition, 2.3.4 (i)***]

$= c[(c\mathcal{A})^{\mathcal{T}_{12}-cl}]$, [by ***proposition 2.3.3 (ii)***]

Hence, $c\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) = c[c\{(c\mathcal{A})^{\mathcal{T}_{12}-Ncl}\}]$

Thus, $c\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) = (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$, since $c[c\mathcal{A}] = \mathcal{A}$.

(ii) By definition we have:

$(c\mathcal{A})^{\mathcal{T}_{12}-Nint} = ((c\mathcal{A})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$= c[\{c((c\mathcal{A})^{\mathcal{T}_2-Nint})\}^{\mathcal{T}_1-Ncl}]$, [by ***proposition 2.3.4 (iii)***]

$= c[\{c\{c((cc\mathcal{A})^{\mathcal{T}_2-Ncl})\}\}^{\mathcal{T}_1-Ncl}]$, [by ***proposition 2.3.4 (iii)***]

$= c[(\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}]$, since $c[c\mathcal{A}] = \mathcal{A}$.

$=c(\mathcal{A}^{\mathcal{T}_{12}-Ncl})$, [by ***proposition 2.3.3 (ii)***]

(iii) By definition we have:

$(\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_{12}-Nint} = ((\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$= [(\mathcal{A}^{\mathcal{T}_2-Nint}) \setminus (\mathcal{B}^{\mathcal{T}_2-Ncl})]^{\mathcal{T}_1-Nint}$, [***proposition 2.3.4 (v)***]

$= ((\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-int}) \setminus ((\mathcal{B}^{\mathcal{T}_2-Ncl}))^{\mathcal{T}_1-Ncl}$,

[***proposition 2.3.4 (v)***]

$= (\mathcal{A}^{\mathcal{T}_{12}-Nint}) \setminus (\mathcal{B}^{\mathcal{T}_{12}-Ncl})$.

(iv) By definition we have:

$(\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_{12}-Ncl} = ((\mathcal{A} \setminus \mathcal{B})^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}$

$= \mathcal{T}_1\text{-}cl((\mathcal{A}^{\mathcal{T}_2-Ncl}) \setminus (\mathcal{B}^{\mathcal{T}_2-Nint}))$, [***proposition 2.3.4 (vi)***]

$= ((\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}) \setminus ((\mathcal{B}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint})$,

[***proposition 2.3.4 (vi)***]

$= (\mathcal{A}^{\mathcal{T}_{12}-Ncl}) \setminus (\mathcal{A}^{\mathcal{T}_{12}-Nint})$.

Corollary 3.3.1

(i) $\mathcal{A}^{\mathcal{T}_{12}-Nint} = c[(c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

(i) $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = c[(c\mathcal{A})^{\mathcal{T}_{12}-Nint}]$

Proof*:*

(i) From *(i)* of ***proposition 3.3.3****, we have:* $c(\mathcal{A}^{\mathcal{T}_{12}-Nint}) = (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$. Taking complements on both sides, we get:

$c[c(\mathcal{A}^{\mathcal{T}_{12}-Nint})] = c[(c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$ from which we get:

$\mathcal{A}^{\mathcal{T}_{12}-Nint} = c[(c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

(ii) Taking complements of both sides of *(ii)* of ***proposition 3.3.3****,* we get:

$c[c(\mathcal{A}^{\mathcal{T}_{12}-Ncl})] = c[(c\mathcal{A})^{\mathcal{T}_{12}-Nint}]$, or, $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = c[(c\mathcal{A})^{\mathcal{T}_{12}-Nint}]$.

3.4 Exterior in Neutro-Bi-topological Spaces

Definition 3.4.1

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *is a N-B-TS and* $\mathcal{A} \subseteq \mathcal{X}$ *then the Nu-Bi-exterior of* $\mathcal{A}$ *is denoted by* $\mathcal{A}^{\mathcal{T}_{12}-Next}$ *and is defined as the Nu-exterior with respect to* $\mathcal{T}_1$ *of the Nu-exterior of* $\mathcal{A}$ *with respect to* $\mathcal{T}_2$*.*

That is: $\mathcal{A}^{\mathcal{T}_{12}-Next} = (\mathcal{A}^{\mathcal{T}_2-Next})^{\mathcal{T}_1-Next}$*, where* $\mathcal{A}^{\mathcal{T}_2-Next} = \cup \{\mathcal{O}_i$*: each* $\mathcal{O}_i$ *is* $\mathcal{T}_2$*-N-OS and* $\mathcal{O}_i \subseteq c\mathcal{A}\}$.

Thus, $(\mathcal{A}^{\mathcal{T}_2-Next})^{\mathcal{T}_1-Next} = \cup \{\mathcal{O}_i$*: each* $\mathcal{O}_i$ *is* $\mathcal{T}_1$*-N-OS and* $\mathcal{O}_i \subseteq c(\mathcal{A}^{\mathcal{T}_2-Next})\}$.

Remark 3.4.1

From the following discussion, it may be observed that in a *N-B-TS*, a direct relation cannot be formed between $\mathcal{A}^{\mathcal{T}_{12}-Next}$ and $\mathcal{A}^{\mathcal{T}_{12}-Nint}$ and also a direct relation between $\mathcal{A}^{\mathcal{T}_{12}-Next}$ and $\mathcal{A}^{\mathcal{T}_{12}-Ncl}$ cannot be established. Some pseudo relations could be established between $\mathcal{A}^{\mathcal{T}_{12}-Next}$ and $\mathcal{A}^{\mathcal{T}_{12}-Nint}$ and between $\mathcal{A}^{\mathcal{T}_{12}-Next}$ and $\mathcal{A}^{\mathcal{T}_{12}-Ncl}$.

By definition, we have:

$\mathcal{A}^{\mathcal{T}_{12}-Next} = (\mathcal{A}^{\mathcal{T}_2-Next})^{\mathcal{T}_1-Next}$

$= ((c\mathcal{A})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Next}$, [*by* ***proposition 2.2.1 (ii)***]

$= [c((c\mathcal{A})^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint}$, [*by* ***proposition 2.2.1 (ii)***]

This shows that there is no direct relation between $\mathcal{A}^{\mathcal{T}_{12}-Next}$and $\mathcal{A}^{\mathcal{T}_{12}-Nint}$.

We may also have:

$(c\mathcal{A})^{\mathcal{T}_{12}-Next} = ((c\mathcal{A})^{\mathcal{T}_2-Next})^{\mathcal{T}_1-Next}$

$= (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nxt}$, using $(c\mathcal{A})^{\mathcal{T}_2-Next} = \{c(c\mathcal{A})\}^{\mathcal{T}_2-Nint}$

$= [c(\mathcal{A}^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint}$*, using* $\mathcal{A}^{\mathcal{T}_1-ext} = (c\mathcal{A})^{\mathcal{T}_1-int}$

Moreover, $\mathcal{A}^{\mathcal{T}_{12}-Nint}$ and $\mathcal{A}^{\mathcal{T}_{12}-Ncl}$ can be related by the relations established in ***proposition 3.3.3*** and the ***corollary 3.3.1***.

The pseudo relation between $\mathcal{A}^{\mathcal{T}_{12}-Next}$ and $\mathcal{A}^{\mathcal{T}_{12}-Nint}$ obtained above is:

$\mathcal{A}^{\mathcal{T}_{12}-Next} = [c((c\mathcal{A})^{\mathcal{T}_2-Nint})]^{\mathcal{T}_1-Nint}$

$= (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Nint}$, [by ***proposition 2.3.4 (iv)***]

$= c[\{c(\mathcal{A}^{\mathcal{T}_2-Ncl})\}^{\mathcal{T}_1-Ncl}]$, [by ***proposition 2.3.4 (iii)***]

Thus, $\mathcal{A}^{\mathcal{T}_{12}-Next} = c[\{c(\mathcal{A}^{\mathcal{T}_2-Ncl})\}^{\mathcal{T}_1-Ncl}]$, which can be treated as a pseudo relation between the two aspects.

Definition 3.4.2

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *be a N-B-TS and* $\mathcal{A} \subseteq \mathcal{X}$*, then the Nu-pseudo-exterior of* $\mathcal{A}$ *denoted by* $\mathcal{A}^{\mathcal{T}_{12}^p-Next}$ *and is defined as* $\mathcal{A}^{\mathcal{T}_{12}^p-Next} = (c\mathcal{A})^{\mathcal{T}_{12}-Nint} = ((c\mathcal{A})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$.

Proposition 3.4.1

For a NBTS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ *if* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$*, then the results that follows hold:*

(i) $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} \subseteq c\mathcal{A}$

(ii) $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} = c\mathcal{A}$*, if* $\mathcal{A}$ *is* $\mathcal{T}_{12}$*-N-CS.*

(iii) $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} \cap \mathcal{A}^{\mathcal{T}_{12}-Nint} = \emptyset$.

(iv) $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (c\mathcal{A})^{\mathcal{T}_{12}^{p}-Next}$

(v) $[c(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next})]^{\mathcal{T}_{12}^{p}-Next} \subseteq \mathcal{A}^{\mathcal{T}_{12}^{p}-Next}$

(vi) *If* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\mathcal{B}^{\mathcal{T}_{12}^{p}-Next} \subseteq \mathcal{A}^{\mathcal{T}_{12}^{p}-Next}$

(vii) $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq (\mathcal{A}^{\mathcal{T}_{12}^{p}-Next})^{\mathcal{T}_{12}^{p}-Next}$

(viii) $(\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}^{p}-Next} \subseteq \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right) \cap \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)$

(ix) $\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}^{p}-Next}\right) \subseteq (\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}^{p}-Next}$

Proof:

(i) We have: $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} = (c\mathcal{A})^{\mathcal{T}_{12}-Nint} \subseteq c\mathcal{A}$, [by ***proposition 3.2.2 (i)***]

(ii) If $\mathcal{A}$ is $\mathcal{T}_{12}$-*N-CS*, then $c\mathcal{A}$ is $\mathcal{T}_{12}$-*N-OS*.

Thus, $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} = (c\mathcal{A})^{\mathcal{T}_{12}-Nint} = c\mathcal{A}$, [by ***proposition 3.2.1***]

(iii) Let $x \in \mathcal{A}^{\mathcal{T}_{12}^{p}-Next} \cap \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}^{p}-Next}$ and $x \in \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in ((c\mathcal{A})^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$ and $x \in (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$\Rightarrow x \in c\mathcal{A}$ and $x \in \mathcal{A}$ which is not possible and so we must have:

$\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} \cap \mathcal{A}^{\mathcal{T}_{12}-Nint} = \emptyset$.

(iv) We have: $(c\mathcal{A})^{\mathcal{T}_{12}^{p}-Next} = (c(c\mathcal{A}))^{\mathcal{T}_{12}-Nint}$

$= \mathcal{A}^{\mathcal{T}_{12}-Nint}$, since $c(c\mathcal{A}) = \mathcal{A}$

(v) We have:

$[c\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)]^{\mathcal{T}_{12}^{p}-Next} = [c((c\mathcal{A})^{\mathcal{T}_{12}-Nint})]^{\mathcal{T}_{12}^{p}-Next}$

$= [c\{c\left((c\mathcal{A})^{\mathcal{T}_{12}-Nint}\right)\}]^{\mathcal{T}_{12}-Nint}$

$= [(c\mathcal{A})^{\mathcal{T}_{12}-Nint}]^{\mathcal{T}_{12}-Nint}$

$\subseteq (c\mathcal{A})^{\mathcal{T}_{12}-Nint}$ [by ***proposition 3.2.2 (iv)***]

$= \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)$, [by definition]

Hence, $[c(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next})]^{\mathcal{T}_{12}^{p}-Next} \subseteq \mathcal{A}^{\mathcal{T}_{12}^{p}-Next}$

(vi) We have: $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c\mathcal{B} \subseteq c\mathcal{A}$, and by ***proposition 3.2.2 (iii)*** we get:

$$(c\mathcal{B})^{\mathcal{T}_{12}-Nint} \subseteq (c\mathcal{A})^{\mathcal{T}_{12}-Nint}$$
$$\Rightarrow \mathcal{B}^{\mathcal{T}_{12}^{p}-Next} \subseteq \mathcal{A}^{\mathcal{T}_{12}^{p}-Next}.$$

(vii) We have by *(i)*, $\mathcal{A}^{\mathcal{T}_{12}^{p}-Next} \subseteq c\mathcal{A}$

$$\Rightarrow \mathcal{A} \subseteq c\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)$$
$$\Rightarrow \mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \{c\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)\}^{\mathcal{T}_{12}-Nint}$$
$$= \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Nxt}\right)^{\mathcal{T}_{12}^{p}-Next}$$

Hence, $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq (\mathcal{A}^{\mathcal{T}_{12}^{p}-Next})^{\mathcal{T}_{12}^{p}-Next}$.

(viii) We have: $(\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}^{p}-Next} = \{c(\mathcal{A} \cup \mathcal{B})\}^{\mathcal{T}_{12}-Nint}$

$$= (c\mathcal{A} \cap c\mathcal{B})^{\mathcal{T}_{12}-Nint}$$
$$\subseteq (c\mathcal{A})^{\mathcal{T}_{12}-Nint} \cap (c\mathcal{B})^{\mathcal{T}_{12}-Nint},$$

[by ***proposition 3.2.2 (v)***]

Hence, $(\mathcal{A} \cup \mathcal{B})^{\mathcal{T}_{12}^{p}-Next} \subseteq \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right) \cap \left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right)$

(ix) We have: $\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}^{p}-Next}\right)$

$$= (c\mathcal{A})^{\mathcal{T}_{12}-Nint} \cup (c\mathcal{B})^{\mathcal{T}_{12}-Nint}$$

$\subseteq (c\mathcal{A} \cup c\mathcal{B})^{\mathcal{T}_{12}-Nint}$, [by ***proposition 3.2.2 (vi)***]

$$= (c\{\mathcal{A} \cap \mathcal{B}\})^{\mathcal{T}_{12}-Nint} = (\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}^{p}-Next}$$

Hence, $\left(\mathcal{A}^{\mathcal{T}_{12}^{p}-Next}\right) \cup \left(\mathcal{B}^{\mathcal{T}_{12}^{p}-Next}\right) \subseteq (\mathcal{A} \cap \mathcal{B})^{\mathcal{T}_{12}^{p}-Next}$

3.5 Boundary in Neutro-Bi-topological Spaces

Definition 3.5.1

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$ then the Nu-Bi-boundary of $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathcal{T}_{12}-bd}$ and is defined as the intersection of the Nu-Bi-closure of the set $\mathcal{A}$ and the Nu-Bi-closure of the complement of $\mathcal{A}$. Thus, $\mathcal{A}^{\mathcal{T}_{12}-Nbd} = \mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$.

Proposition 3.5.1

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A} \subseteq \mathcal{X}$, then we have the following results:

(i) $\mathcal{A}^{\mathcal{T}_{12}-Nbd} = (c\mathcal{A})^{\mathcal{T}_{12}-Nbd}$

(ii) $\mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint} = \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

(iii) $\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cup ((c\mathcal{A})^{\mathcal{T}_{12}-Nint}) = c(\mathcal{A}^{\mathcal{T}_{12}-Nbd})$

(iv) $\mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd} = \mathcal{A}^{\mathcal{T}_{12}-Nint}$

(v) $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right) \cup (\mathcal{A}^{\mathcal{T}_{12}-Nbd})$

Proof*:*

(i) We have by definition:

$$(c\mathcal{A})^{\mathcal{T}_{12}-Nbd} = (c\mathcal{A})^{\mathcal{T}_{12}-Ncl} \cap (c\{c\mathcal{A}\})^{\mathcal{T}_{12}-Ncl}$$
$$= (c\mathcal{A})^{\mathcal{T}_{12}-Ncl} \cap \mathcal{A}^{\mathcal{T}_{12}-Ncl}$$
$$= \mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$$
$$= \mathcal{A}^{\mathcal{T}_{12}-Nbd}$$

(ii) Let $x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$and $x \notin \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$and $x \notin \mathcal{A}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$and $x \in c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$and $x \in (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Hence, $\mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Conversely, *let* $x \in \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Then, $x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ *and* $x \in (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ *and* $x \in c(\mathcal{A}^{\mathcal{T}_{12}-Nint})$, [by ***proposition 3.3.3 (i)***]

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ *and* $x \notin \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint}$

Hence, $\mathcal{A}^{\mathcal{T}_{12}-Nbd} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint}$

Thus, $\mathcal{A}^{\mathcal{T}_{12}-Ncl} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nint} = \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

(iii) We have: $c[\mathcal{A}^{\mathcal{T}_{12}-Nbd}] = c[\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

$$= c[\mathcal{A}^{\mathcal{T}_{12}-Ncl}] \cup c[(c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$$
$$= [(c\mathcal{A})^{\mathcal{T}_{12}-Nint}] \cup (\mathcal{A}^{\mathcal{T}_{12}-Nint}),$$

[by ***proposition 3.3.3 (ii)*** and ***corollary 3.3.1 (i)***]

Thus, $c[\mathcal{A}^{\mathcal{T}_{12}-Nbd}] = \left[(c\mathcal{A})^{\mathcal{T}_{12}-Nint}\right] \cup \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right)$

(iv) For every $x \in \mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

We have $x \in \mathcal{A}$ but $x \notin \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Or, $x \in \mathcal{A}$ but $x \notin [\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

Or, $x \in \mathcal{A}$ but $[x \notin \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ but $x \notin (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

Or, $x \in \mathcal{A}$ but $x \notin \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ but $x \notin c\left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right)$, ***by proposition 3.3.3 (i)***.

Or, $x \in \mathcal{A}$ but $x \notin \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ but $x \in \left(\mathcal{A}^{\mathcal{T}_{12}-Nint}\right)$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{12}-Nint}$ and hence $\mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Nint}$

Conversely, let $x \in \mathcal{A}^{\mathcal{T}_{12}-Nint}$

$\Rightarrow x \in \mathcal{A} \Rightarrow x \in \mathcal{A}$ but $x \notin c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}$ but, $[x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ and $x \notin (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$, since $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

$\Rightarrow x \in \mathcal{A}$ but $x \notin [\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$

$\Rightarrow x \in \mathcal{A}$ but $x \notin \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

$\Rightarrow x \in \mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Hence, $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd}$

Thus, $\mathcal{A} \setminus \mathcal{A}^{\mathcal{T}_{12}-Nbd} = \mathcal{A}^{\mathcal{T}_{12}-Nint}$

(v) Let $x \in [\mathcal{A}^{\mathcal{T}_{12}-Nint}] \cup [\mathcal{A}^{\mathcal{T}_{12}-Nbd}]$

Then, $x \in [\mathcal{A}^{\mathcal{T}_{12}-Nint}]$ or, $x \in [\mathcal{A}^{\mathcal{T}_{12}-Nbd}]$

If we consider the first option, then $x \in [\mathcal{A}^{\mathcal{T}_{12}-Nint}] \subseteq \mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

Hence, $[\mathcal{A}^{\mathcal{T}_{12}-Nint}] \cup [\mathcal{A}^{\mathcal{T}_{12}-Nbd}] \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

If we consider the second option, then $x \in [\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}]$ which gives us: $x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ and $x \in (c\mathcal{A})^{\mathcal{T}_{12}-Ncl}$

Thus, in either case:

$[\mathcal{A}^{\mathcal{T}_{12}-Nint}] \cup [\mathcal{A}^{\mathcal{T}_{12}-Nbd}] \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$.

Conversely, let $x \in \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

Then, $x \in (\mathcal{A}^{\mathcal{T}_{2}-Ncl})^{\mathcal{T}_{1}-Ncl}$

$\Rightarrow x \in \mathcal{A}^{\mathcal{T}_{1}-Ncl}$ and $x \in \mathcal{A}^{\mathcal{T}_{2}-Ncl}$

Now, $x \in \mathcal{A}^{\mathcal{T}_{1}-Ncl} \Rightarrow x \in [\mathcal{A}^{\mathcal{T}_{1}-Nint} \cup \mathcal{A}^{\mathcal{T}_{1}-Nbd}]$, [***proposition 2.4.1 (vi)***]

And, $x \in \mathcal{A}^{\mathcal{T}_{2}-Ncl} \Rightarrow x \in [\mathcal{A}^{\mathcal{T}_{2}-Nint} \cup \mathcal{A}^{\mathcal{T}_{2}-Nbd}]$

Hence, $x \in [\mathcal{A}^{\mathcal{T}_{12}-Nint} \cup \mathcal{A}^{\mathcal{T}_{12}-Nbd}]$

Hence, $\mathcal{A}^{\mathcal{T}_{12}-Ncl} \subseteq [\mathcal{A}^{\mathcal{T}_{12}-Nint} \cup \mathcal{A}^{\mathcal{T}_{12}-Nbd}]$

Thus, $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = [\mathcal{A}^{\mathcal{T}_{12}-Nint} \cup \mathcal{A}^{\mathcal{T}_{12}-Nbd}]$

3.6 Neutro-Quasi Open and Closed Sets

Definition 3.6.1

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$, then the subset $\mathcal{A}$ will be called Nu-Quasi-Open, written as N-QO, if for each $x \in \mathcal{A}$, there exists a N-OS $\mathcal{Q}$, N-O with respect to either $\mathcal{T}_1$ or $\mathcal{T}_2$, so that $x \in \mathcal{Q} \subseteq \mathcal{A}$.

A set is Nu-Quasi-Closed, written as N-QC if its complement is N-QO.

Remark 3.6.1

Every *N-OS* is *N-QO* but converse may not be always true. Same holds for *N-QC* sets.

Definition 3.6.2

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$, then the Nu-quasi-interior of $\mathcal{A}$, denoted by $\mathcal{A}^{NQ-int}$, is the union of all N-QO sets which are subsets of $\mathcal{A}$.

Thus, $\mathcal{A}^{NQ-int} = \cup \{\mathcal{O}_i$: each $\mathcal{O}_i$ is NQO and each $\mathcal{O}_i \subseteq \mathcal{A}\}$.

Proposition 3.6.1

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$. Let $\mathcal{A}, \mathcal{B}$ be N-QO sets, then $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B}$ are also N-QO sets.

Proposition 3.6.2

If $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ is a NBTS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the results that follow are true:

(i) $\mathcal{A}^{NQ-int} \subseteq \mathcal{A}$

(ii) $\mathcal{A}^{NQ-int} = \mathcal{A}$, *if $\mathcal{A}$ is N-QO.*

(iii) $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{NQ-int}$

(iv) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{NQ-int} \subseteq \mathcal{B}^{NQ-int}$

(v) $(\mathcal{A} \cap \mathcal{B})^{NQ-int} = (\mathcal{A}^{NQ-int}) \cap (\mathcal{B}^{NQ-int})$

(vi) $(\mathcal{A}^{NQ-int}) \cup \left(\mathcal{B}^{NQ-int}\right) \subseteq (\mathcal{A} \cup \mathcal{B})^{NQ-int}$

Proof:

(i) We have $\mathcal{A}^{NQ-int} \subseteq \mathcal{A}$, by definition.

(ii) If $\mathcal{A}$ is *N-QO*, then for every $x \in \mathcal{A}$, there exists *N-OS* $\mathcal{O}_x$ so that $\mathcal{O}_x$ is *N-OS* with respect to either $\mathcal{T}_1$ or $\mathcal{T}_2$ such that $x \in \mathcal{O}_x \subseteq \mathcal{A}$.

Now, if $x \in \mathcal{A}^{NQ-int}$, then $x \in \mathcal{Q}_x$, where $\mathcal{Q}_x$ is *N-QO* and the same will be the case for every $x \in \mathcal{A}$ and by ***proposition 3.6.1*** since $\mathcal{A}^{NQ-int}$ is the union of *N-QO* sets which are subsets of $\mathcal{A}$ so $\mathcal{A}^{NQ-int}$ will also be a *N-QO* set and thus: $\cup\{Q_x : x \in \mathcal{A}\} = \mathcal{A}$.

(iii) We have: $\mathcal{A}^{\mathcal{T}_{12}-Nint} = (\mathcal{A}^{\mathcal{T}_2-Nint})^{\mathcal{T}_1-Nint}$

$$=\cup\{\mathcal{O}_i : \text{each } \mathcal{O}_i \text{ is } \mathcal{T}_1\text{-}N\text{-}OS \text{ and } \mathcal{O}_i \subseteq \mathcal{A}^{\mathcal{T}_2-Nint}\}$$

This can be a null set if there is no such subset which is $\mathcal{T}_1$-*N-OS*.

However, $\mathcal{A}^{NQ-int} =\cup\{\mathcal{O}_i$: each $\mathcal{O}_i$ is *N-QO* and each $\mathcal{O}_i \subseteq \mathcal{A}\} \neq \emptyset$ because a subset is *N-QO* if and only if for every element that belongs to it, there is always a *N-OS* in either of the two *N-Ts* which is contained in the subset.

Thus, $\mathcal{A}^{\mathcal{T}_{12}-Nint} \subseteq \mathcal{A}^{NQ-int}$

(iv) We have $\mathcal{A}^{NQ-int} =\cup\{\mathcal{O}_i$: each $\mathcal{O}_i$ is *N-QO* and each $\mathcal{O}_i \subseteq \mathcal{A}\}$

$\subseteq\cup\{\mathcal{O}_i$: each $\mathcal{O}_i$ is *N-QO* and each $\mathcal{O}_i \subseteq \mathcal{B}\} = \mathcal{B}^{NQ-int}$,

since $\mathcal{A} \subseteq \mathcal{B}$ and the number of *N-QO* sets may be more in the later union.

(v) For every $x \in (\mathcal{A} \cap \mathcal{B})^{NQ-int}, x \in \mathcal{O}_x$ so that $\mathcal{O}_x$ is *N-QO* and $\mathcal{O}_x \subseteq \mathcal{A} \cap \mathcal{B}$ which gives $\mathcal{O}_x \subseteq \mathcal{A}$ and $\mathcal{O}_x \subseteq \mathcal{B}$ which in turn shows that $\mathcal{O}_x \subseteq \mathcal{A}^{NQ-int}$ and $\mathcal{O}_x \subseteq \mathcal{B}^{NQ-int}$ thereby showing that $\mathcal{O}_x \subseteq \mathcal{A}^{NQ-int} \cap \mathcal{B}^{NQ-int}$ and hence we have: $(\mathcal{A} \cap \mathcal{B})^{NQ-int} \subseteq (\mathcal{A}^{NQ-int}) \cap (\mathcal{B}^{NQ-int})$.

Conversely, for every $x \in (\mathcal{A}^{NQ-int}) \cap (\mathcal{B}^{NQ-int})$, we have $x \in Q_x$ where Q_x is *N-QO* and $Q_x \subseteq \mathcal{A}^{NQ-int} \cap \mathcal{B}^{NQ-int}$ which shows that $Q_x \subseteq \mathcal{A}^{\mathrm{NQ-int}}$ and $Q_x \subseteq \mathcal{B}^{NQ-int}$ which in turn implies that $Q_x \subseteq \mathcal{A}$ and $Q_x \subseteq \mathcal{B}$ implying that $Q_x \subseteq \mathcal{A} \cap \mathcal{B}$.

Thus, $x \in Q_x \subseteq \mathcal{A} \cap \mathcal{B}$ which implies that $x \in Q_x$ and Q_x is *N-QO* and $Q_x \subseteq \mathcal{A} \cap \mathcal{B}$ thereby showing that $x \in (\mathcal{A} \cap \mathcal{B})^{NQ-int}$

Hence, $(\mathcal{A} \cap \mathcal{B})^{NQ-int} = (\mathcal{A}^{NQ-int}) \cap (\mathcal{B}^{NQ-int})$

(vi) For every $x \in (\mathcal{A}^{NQ-int}) \cup (\mathcal{B}^{NQ-int})$ we have $x \in \mathcal{A}^{NQ-int}$ or, $x \in \mathcal{B}^{NQ-int}$ which gives: $x \in \mathcal{P}_x \subseteq \mathcal{A}$ or $x \in Q_x \subseteq \mathcal{B}$ where $\mathcal{P}_x$ and Q_x are *N-QO* sets. This, in turn, shows that $x \in \mathcal{P}_x \cup Q_x \subseteq \mathcal{A} \cup \mathcal{B}$.

Now, union of two *N-QO* sets, by ***proposition 6.1***, is *N-QO*, thereby showing that $x \in (\mathcal{A} \cup \mathcal{B})^{NQ-int}$.

Hence, we get: $(\mathcal{A}^{NQ-int}) \cup (\mathcal{B}^{NQ-int}) \subseteq (\mathcal{A} \cup \mathcal{B})^{NQ-int}$

Remark 3.6.2

Equality will hold in ***(v)*** above and it is because by ***proposition 3.6.1***, every *N-OS* is *N-QO*.

Definition 3.6.3

Let $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$ be a N-B-TS and $\mathcal{A} \subseteq \mathcal{X}$, then the subset $\mathcal{A}$ will be called Neutro-pseudo-open, written as N-PO, if it is a N-OS in $\mathcal{T}_1 \cup \mathcal{T}_2$.

Remark 3.6.3

Every *N-PO* set is *N-QO* but the converse is not necessarily true.

Definition 3.6.4

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A} \subseteq \mathcal{X}$, then the Nu-quasi-closure of $\mathcal{A}$, denoted by $\mathcal{A}^{NQ-Cl}$ and is defined as $\mathcal{A}^{NQ-Cl} = \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl}$.

Proposition 3.6.3

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, and $\mathcal{A} \subseteq \mathcal{X}$, if $\mathcal{A}$ is $\mathcal{T}_{12}$-N-CS then $\mathcal{A}^{NQ-Cl} = \mathcal{A}$.

Proof:

If $\mathcal{A}$ is *NCS* then by ***proposition 2.3.2*** $\mathcal{A}^{\mathcal{T}_1-Ncl} = \mathcal{A}$ and $\mathcal{A}^{\mathcal{T}_2-Ncl} = \mathcal{A}$ and hence $\mathcal{A}^{NQ-Cl} = \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl} = \mathcal{A} \cap \mathcal{A} = \mathcal{A}$.

Proposition 3.6.4

For a N-B-TS $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$ then the results that follows are true.

- *(i)* $\mathcal{A}^{NQ-Cl} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$
- *(ii)* $\mathcal{A} \subseteq \mathcal{A}^{NQ-Cl}$
- *(iii)* $(\mathcal{A}^{NQ-Cl})^{NQ-Cl} \subseteq \mathcal{A}^{NQ-Cl}$
- *(iv)* *If* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{A}^{NQ-Cl} \subseteq \mathcal{B}^{NQ-Cl}$

Proof:

(i) We have $\mathcal{A}^{NQ-Cl} = \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl}$ and,

$\mathcal{A}^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} = \cup \{\mathcal{C}_i$: each $\mathcal{C}_i$ is $\mathcal{T}_1$-*NCS* and $\mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{C}_i\}$

Now, if $\mathcal{A}^{\mathcal{T}_2-Ncl} \subset \mathcal{X}$, then $\mathcal{A}^{NQ-Cl} = \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl} \subset \mathcal{X}$, whatever the value of $\mathcal{A}^{\mathcal{T}_1-Ncl}$ be.

But $\mathcal{A}^{\mathcal{T}_{12}-Ncl} = (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} = \mathcal{X}$, *if* $\mathcal{A}^{\mathcal{T}_2-Ncl} \subset \mathcal{X}$ but no such $\mathcal{C} \subset \mathcal{X}$ exist so that $\mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{C}$. Hence $\mathcal{A}^{NQ-Cl} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$ in general *(ii)* From ***proposition 3.3.2 (i)*** $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_1-Ncl}$ and $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_2-Ncl}$ and hence we have $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl} = \mathcal{A}^{NQ-Cl}$

(iii) We have: $(\mathcal{A}^{NQ-Cl})^{NQ-Cl}$

$= (\mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl})^{NQ-Cl}$

$= (\mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl} \cap (\mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_2-Ncl}$

$\subseteq [(\mathcal{A}^{\mathcal{T}_1-Ncl})^{\mathcal{T}_1-Ncl} \cap (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}] \cap [(\mathcal{A}^{\mathcal{T}_1-Ncl})^{\mathcal{T}_2-Ncl} \cap$

$(\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_2-Ncl}]$ [by ***proposition 2.3.3 (v)***]

$= [\mathcal{A}^{\mathcal{T}_1-Ncl} \cap (\mathcal{A}^{\mathcal{T}_2-Ncl})^{\mathcal{T}_1-Ncl}] \cap [(\mathcal{A}^{\mathcal{T}_1-Ncl})^{\mathcal{T}_2-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl}]$

[by ***proposition 2.3.3 (ii)***]

$= [\mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_{12}-Ncl}] \cap [\mathcal{A}^{\mathcal{T}_{12}-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl}]$

$= \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl} \cap \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

$= \mathcal{A}^{NQ-Cl} \cap \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

$= \mathcal{A}^{NQ-Cl}$, *by (i)* since $\mathcal{A}^{NQ-Cl} \subseteq \mathcal{A}^{\mathcal{T}_{12}-Ncl}$

Hence, $(\mathcal{A}^{NQ-Cl})^{NQ-Cl} \subseteq \mathcal{A}^{NQ-Cl}$

(iv) We have: $\mathcal{A}^{NQ-Cl} = \mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl}$

Now, $\mathcal{A}^{\mathcal{T}_1-Ncl} \subseteq \mathcal{B}^{\mathcal{T}_1-Ncl}$ and $\mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{B}^{\mathcal{T}_2-Ncl}$ if $\mathcal{A} \subseteq \mathcal{B}$,

[***proposition 2.3.3 (iii)***]

Hence, $\mathcal{A}^{\mathcal{T}_1-Ncl} \cap \mathcal{A}^{\mathcal{T}_2-Ncl} \subseteq \mathcal{B}^{\mathcal{T}_1-Ncl} \cap \mathcal{B}^{\mathcal{T}_2-Ncl}$

Or, $\mathcal{A}^{NQ-Cl} \subseteq \mathcal{B}^{NQ-Cl}$