CHAPTER 4

Study on Interior, Exterior, Closure, and Boundary in Anti-Topological Spaces

In this chapter the aspects of interior, exterior, closure, and boundary are defined in anti-topological spaces (A-TS) and the various properties of these aspects that are generally true for GTS are inspected. Whenever certain properties that are valid in GTS are found to be not holding in the A-TS, those results are substituted by similar results with different conditions, like if equality does not hold for a certain property, then it is verified whether containment is satisfied and if true such results with equality are replaced by results with containment.

4.0 Some basic concepts in Anti-Topological Spaces

Proposition 4.0.1

For a non-empty set $\mathcal{X}$ and a collection of subsets $\mathcal{T}$ of $\mathcal{X}$, referred to as anti-open sets (A-OS), $\mathcal{T}$ is an anti-topology (A-T) and $(\mathcal{X}, \mathcal{T})$ an A-TS if all of the following are true:

- *(i) The null-set and the whole set are not in $\mathcal{T}$.*
- *(ii) There union of members of $\mathcal{T}$ are not in $\mathcal{T}$.*
- *(iii) There intersection of members of $\mathcal{T}$ are not in $\mathcal{T}$.*

Remark 4.0.1

The union of *A-Ts* need not necessarily be an *A-T*. It can be seen by the following example: If we assume $\mathcal{X} = \{1,2,3,4,5\}$ and consider $\mathcal{T}_1 = \big\{\{1\}, \{2\big\}, \{3\}, \{4\}, \{5\}\}$ and $\mathcal{T}_2 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ to be two *ATs* on the set $\mathcal{X}$.

Then, $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\big\{\{1\}, \{2\big\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ is not an *A-T* on $\mathcal{X}$ because the collection defies ***proposition 4.0.1*** as $\{1\}, \{2\}, \{1,2\} \in \mathcal{T}_1 \cup \mathcal{T}_2$.

Some of the results discussed in this chapter have been published in: Basumatary, B., & Khaklary, J.K. (2022). A Study on the Properties of Anti-Topological Spaces. *Neutrosophic Algebraic Structures and their Applications, (pp. 16-27), IGI Global.*

The intersection of two *A-Ts* on a set need not be an *A-T* on the set because if we consider the above *A-Ts*, we have $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ and it is not an *A-T*.

Proposition 4.0.2

For an A-TS $(\mathcal{X}, \mathcal{T})$ *and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, *the following properties are synonymous with* $\mathcal{T}$:

(i) $\mathcal{A}, \mathcal{B} \notin \mathcal{T}$ *whenever.* $\mathcal{A} \cup \mathcal{B} \in \mathcal{T}$

(ii) $\mathcal{A}, \mathcal{B} \notin \mathcal{T}$ *whenever* $\mathcal{A} \cap \mathcal{B} \in \mathcal{T}$

(iii) $\mathcal{B} \notin \mathcal{T}$ *whenever* $\mathcal{A} \in \mathcal{T}$ *and* $\mathcal{B} \subseteq \mathcal{A}$

4.1 Interior in Anti-Topological Spaces

Definition 4.1.1

Let $(\mathcal{X}, \mathcal{T})$ *be an A-TS on the set* $\mathcal{X}$ *and* $\mathcal{A}$ *be a proper subset of* $\mathcal{X}$, *the anti-interior of* $\mathcal{A}$ *is defined to be union of the subsets of* $\mathcal{A}$ *which are A-O and denoted by* $\mathcal{A}^{Anti-int}$.

That is, $\mathcal{A}^{Anti-int} = \cup \{\mathcal{O}_i : \mathcal{O}_i \subseteq \mathcal{A}$ *and each* $\mathcal{O}_i$ *is A-O*$\}$.

We define: $\emptyset^{Anti-int} = \emptyset$.

Remark 4.1.1

In an *A-TS*, the null set is not anti-open and as such there will be instances of non-existences of *A-O* subsets while trying to find the anti-interior of certain sets. If no *A-O* subsets exist for a particular set, we will conclude that the anti-interior of that set does not exist with respect to the *A-T* in context. However, in trying to establish results for the anti-interior in the few propositions that follows, it has been assumed that the anti-interior exists for the subsets we have considered. In other words, it has been suggested that cases of non-existences of anti-interiors are ignored.

Proposition 4.1.1

Let $(\mathcal{X}, \mathcal{T})$ *be an A-TS on the set* $\mathcal{X}$ *and* $\mathcal{A} \subseteq \mathcal{X}$, *then if* $\mathcal{A}$ *is A-O then* $\mathcal{A}^{Anti-int} = \mathcal{A}$.

Remark 4.1.2

In an *A-TS* $\mathcal{X}$ and for any $\mathcal{A} \subset \mathcal{X}$, $\mathcal{A}^{Anti-int}$ need not be the largest *A-O* subset contained in $\mathcal{A}$. This fact is because, in an *A-TS*, the union of *A-OSs* is not *A-O*. The following example may be considered:

Consider $\mathcal{X} = \{1,2,3,4,5\}$ and $\mathcal{T} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$ and consider $\mathcal{A} = \{2,3,4\}$ then $\mathcal{A}^{Anti-int} = \{2,3\} \cup \{2,4\} = \{2,3,4\} = \mathcal{A}$. However, $\mathcal{A}$ is not *A-O*.

The above example also shows that ***proposition 4.1.1*** is not always true the other way around. That is if $\mathcal{A}^{Anti-int} = \mathcal{A}$, then $\mathcal{A}$ need not be necessarily *A-O*.

Proposition 4.1.2

If $(\mathcal{X}, \mathcal{T})$ *is an A-TS and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$ *then the following results are true:*

(i) $\mathcal{A}^{Anti-int} \subseteq \mathcal{A}$.

(ii) $(\mathcal{A}^{Anti-int})^{Anti-int} = \mathcal{A}^{Anti-int}$.

(iii) *If* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{A}^{Anti-int} \subseteq \mathcal{B}^{Anti-int}$.

(iv) $(\mathcal{A} \cap \mathcal{B})^{Anti-int} \subseteq \mathcal{A}^{Anti-int} \cap \mathcal{B}^{Anti-int}$.

(v) $\mathcal{A}^{Anti-int} \cup \mathcal{B}^{Anti-int} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-int}$.

Proof:

(i) Consider an element x that belongs to the anti-interior of $\mathcal{A}$ then x is contained in some *A-OS* $\mathcal{B}_i$ which are all subsets of the set $\mathcal{A}$ and hence the result.

(ii) Let $\mathcal{A}^{Anti-int} = \mathcal{O} = \cup \{\mathcal{O}_i$: each $\mathcal{O}_i$ is *A-OS* and each $\mathcal{O}_i \subseteq \mathcal{A}\}$, then

$(\mathcal{A}^{Anti-int})^{Anti-int} = (\mathcal{O})^{Anti-int}$

$= \cup \{\mathcal{O}_i$:each$\mathcal{O}_i$is *A-OS*, each$\mathcal{O}_i \subseteq \mathcal{A}\}$

$= \mathcal{O} = \mathcal{A}^{Anti-int}$

(iii) We have by *(i)* $\mathcal{A}^{Anti-int} \subseteq \mathcal{A} \subseteq \mathcal{B}$ and hence $\mathcal{A}^{Anti-int} \subseteq \mathcal{B}$. Now, $\mathcal{A}^{Anti-int}$ is a union of *A-OSs* which are contained in $\mathcal{B}$ and so it will either be the anti-interior of $\mathcal{B}$ or contained in the anti-interior of $\mathcal{B}$.

That is, $\mathcal{A}^{Anti-int} = \mathcal{B}^{Anti-int}$ or $\mathcal{A}^{Anti-int} \subseteq \mathcal{B}^{Anti-int} \subseteq \mathcal{B}$.

In either case, $\mathcal{A}^{Anti-int} \subseteq \mathcal{B}^{Anti-int}$ if $\mathcal{A} \subseteq \mathcal{B}$.

(iv) We have $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$ and using *(iii)* above will give:

$(\mathcal{A} \cap \mathcal{B})^{Anti-int} \subseteq \mathcal{A}^{Anti-int}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B}$ with *(iii)* gives:

$(\mathcal{A} \cap \mathcal{B})^{Anti-int} \subset \mathcal{B}^{Anti-int}$

Hence $(\mathcal{A} \cap \mathcal{B})^{Anti-int} \subseteq \mathcal{A}^{Anti-int} \cap \mathcal{B}^{Anti-int}$.

(v) Let $x \in \mathcal{A}^{Anti-int} \cup \mathcal{B}^{Anti-int}$

$\Rightarrow x \in \mathcal{A}^{Anti-int}$ or, $x \in \mathcal{B}^{Anti-int}$

$\Rightarrow x \in \mathcal{O}_{\mathcal{A}} \subseteq \mathcal{A}$ or, $x \in \mathcal{O}_{\mathcal{B}} \subseteq \mathcal{B} \Rightarrow x \in \mathcal{O}_{\mathcal{A}} \cup \mathcal{O}_{\mathcal{B}} \subseteq \mathcal{A} \cup \mathcal{B}$

$\Rightarrow x \in (\mathcal{A} \cup \mathcal{B})^{Nt-int}$ and hence $\mathcal{A}^{Anti-int} \cup \mathcal{B}^{Anti-int} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-int}$.

Remark 4.1.3

In the case of *GTS* equality holds in case of the result *(iv)* which however is not the case in *A-TS*, and the following example illustrates the case. Consider $\mathcal{X} = \{1,2,3,4,5\}$ and an let $\mathcal{T} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{3,4\},\{4,5\}$, then $(\mathcal{X},\mathcal{T})$ is an *A-TS*. Let $\mathcal{A} = \{2,3,4,5\}$ and $\mathcal{B} = \{1,2,3,5\}$ then $\mathcal{A} \cap \mathcal{B} = \{2,3,5\}$ and $(\mathcal{A} \cap \mathcal{B})^{Anti-int} = \{2,3\}$. Also, $\mathcal{A}^{Anti-int} = \{2,3,4,5\}$ and $\mathcal{B}^{Anti-int} = \{1,2,3,5\}$, so $\mathcal{A}^{Anti-int} \cap \mathcal{B}^{Anti-int} = \{2,3,5\}$. Thus, $(\mathcal{A} \cap \mathcal{B})^{Anti-int} \neq \mathcal{A}^{Anti-int} \cap \mathcal{B}^{Anti-int}$ in general in the case of an *A-TS*.

Remark 4.1.4

The reason for ***proposition 4.1.2 (iv)*** not holding for the equality sign as in the case of *GTS* is because intersection of *A-OSs* in an *A-TS* is not *A-O*.

Definition 4.1.2

For an A-TS $(\mathcal{X},\mathcal{T})$ and $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$, the anti-interior operator on the space $\mathcal{X}$ is a function $Anti-int$: $\mathcal{I}(\mathcal{X}) \rightarrow \mathcal{I}(\mathcal{X})$ such that:

(i) $\mathcal{A}^{Anti-int} \subseteq \mathcal{A}$

(ii) $\left(\mathcal{A}^{Anti-int}\right)^{Anti-int} = \mathcal{A}^{Anti-int}$

(iii) $(\mathcal{A} \cap \mathcal{B})^{Anti-int} \subseteq \left(\mathcal{A}^{Anti-int}\right) \cap \left(\mathcal{B}^{Anti-int}\right)$

(iv) $(\mathcal{A} \cup \mathcal{B})^{Anti-int} \subseteq \left(\mathcal{A}^{Anti-int}\right) \cup \left(\mathcal{B}^{Anti-int}\right)$

4.2 Exterior in Anti-Topological Spaces

Definition 4.2.1

For an A-TS $(\mathcal{X},\mathcal{T})$ and $\mathcal{A} \subseteq \mathcal{X}$, the anti-exterior of $\mathcal{A}$ is defined as the union of subsets of $c\mathcal{A}$ which are A-O and is denoted by $\mathcal{A}^{Anti-ext}$. That is, $\mathcal{A}^{Anti-ext} = \cup \{\mathcal{O}_i : \mathcal{O}_i \subseteq c\mathcal{A}$ *and each* $\mathcal{O}_i$ *is A-O*$\}$. *We define:* $\mathcal{X}^{Anti-ext} = \emptyset$ *and* $\emptyset^{Anti-ext} = \mathcal{X}$.

Remark 4.2.1

$\mathcal{A}^{Anti-ext}$ is the union of all subsets of the A-T that do not intersect $\mathcal{A}$. Thus, $\mathcal{A}^{Anti-ext}$ is larger than any other A-OS that do not intersect $\mathcal{A}$.

Proposition 4.2.1

Let $(\mathcal{X},\mathcal{T})$ be an A-TS on the set $\mathcal{X}$ and $\mathcal{A},\mathcal{B} \subseteq \mathcal{X}$, then the following are true:

(i) $\mathcal{A}^{Anti-ext} \subseteq c\mathcal{A}$

(ii) $\mathcal{A}^{Anti-ext} = (c\mathcal{A})^{Anti-int}$

(iii) $\mathcal{A}^{Anti-ext} = [c(\mathcal{A}^{Anti-ext})]^{Anti-ext}$

(iv) $\mathcal{A}^{Anti-int} = (c\mathcal{A})^{Anti-ext}$

(v) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathcal{A}^{Anti-ext} \supseteq \mathcal{B}^{Anti-ext}$

(vi) $(\mathcal{A}^{Ant-ext})^{Anti-ext} \supseteq \mathcal{A}^{Anti-int}$

(vii) $(\mathcal{A} \cup \mathcal{B})^{Anti-ext} \subseteq \mathcal{A}^{Anti-ext} \cap \mathcal{B}^{Anti-ext}$

(viii) $\mathcal{A}^{Anti-ext} \cup \mathcal{B}^{Anti-ext} \subseteq (\mathcal{A} \cap \mathcal{B})^{Anti-ext}$

(ix) $\mathcal{A}^{Anti-ext} = c\mathcal{A}$ *if* $\mathcal{A}$ *is Anti-closed (A-C)*

(x) $\mathcal{A}^{Anti-int} \cap \mathcal{A}^{Anti-ext} = \emptyset$

Proof:

(i) $\mathcal{A}^{Anti-ext} = (c\mathcal{A})^{Anti-int} \subseteq c\mathcal{A}$ by ***proposition 4.1.2 (i)***.

(ii) By definition*:* $\mathcal{A}^{Anti-ext} = \cup \{\mathcal{O}_i : \mathcal{O}_i \subseteq c\mathcal{A} \text{ and each } \mathcal{O}_i \text{ is A-O}\}$

$= (c\mathcal{A})^{Anti-int}$

(iii) We have:

$[c(\mathcal{A}^{Anti-ext})]^{Anti-ext} = [c(c\mathcal{A})^{Anti-int}]^{Anti-ext}$, *by (ii).*

$= [c\{c(c\mathcal{A})^{Anti-int}\}]^{Anti-int}$, *by (ii)*

$= [(c\mathcal{A})^{Anti-int}]^{Anti-int}$, $c(c\mathcal{A}) = \mathcal{A}$.

$= (c\mathcal{A})^{Anti-int}$, $(\mathcal{A}^{Anti-int})^{Anti-int} = \mathcal{A}^{Anti-int}$

$= \mathcal{A}^{Anti-ext}$

(iv) We have: $(c\mathcal{A})^{Anti-ext} = \cup \{\mathcal{O}_i : \mathcal{O}_i \subseteq c(c\mathcal{A}) \text{ and each } \mathcal{O}_i \text{ is A-O}\}$

$= \cup \{\mathcal{O}_i : \mathcal{O}_i \subseteq \mathcal{A} \text{ and each } \mathcal{O}_i \text{ is A-O}\}$

$= \mathcal{A}^{Anti-int}$.

(v) We have $\mathcal{A} \subseteq \mathcal{B} \Rightarrow c\mathcal{B} \subseteq c\mathcal{A}$

$\Rightarrow (c\mathcal{B})^{Anti-int} \subseteq (c\mathcal{A})^{Anti-int}$ by ***proposition 4.1.2 (iii)***

$\Rightarrow \mathcal{B}^{Anti-ext} \subseteq \mathcal{A}^{Anti-ext}$

(vi) By *(i)* $\mathcal{A}^{Anti-ext} \subseteq c\mathcal{A}$ and by *(v)* we have:

$(\mathcal{A}^{Anti-ext})^{Anti-ext} \supseteq (c\mathcal{A})^{Anti-ext} = (cc\mathcal{A})^{Anti-int} = \mathcal{A}^{Anti-int}$.

Thus, $(\mathcal{A}^{Anti-ext})^{Anti-ext} \supseteq \mathcal{A}^{Anti-int}$

(vii) We have:

$(\mathcal{A} \cup \mathcal{B})^{Anti-ext} = (c(\mathcal{A} \cup \mathcal{B}))^{Anti-int}$

$= (c\mathcal{A} \cap c\mathcal{B})^{Anti-int}$

$\subseteq (c\mathcal{A})^{Anti-int} \cap (c\mathcal{B})^{Anti-int}$, [by ***proposition 4.1.2 (iv)***]

$= \mathcal{A}^{Anti-ext} \cap \mathcal{A}^{Anti-ext}$

Thus, $(\mathcal{A} \cup \mathcal{B})^{Anti-ext} \subseteq \mathcal{A}^{Anti-ext} \cap \mathcal{B}^{Anti-ext}$.

(viii) $\mathcal{A}^{Anti-ext} \cup \mathcal{B}^{Anti-ext} = (c\mathcal{A})^{Anti-int} \cup (c\mathcal{B})^{Anti-int}$

$\subseteq (c\mathcal{A} \cup c\mathcal{B})^{Anti-int}$ [by ***proposition 4.1.2 (v)***]

$= (c(\mathcal{A} \cap \mathcal{B}))^{Anti-int}$

$= (\mathcal{A} \cap \mathcal{B})^{Anti-ext}$.

Thus, $\mathcal{A}^{Anti-ext} \cup \mathcal{B}^{Anti-ext} \subseteq (\mathcal{A} \cap \mathcal{B})^{Anti-ext}$

(ix) We have $\mathcal{A}^{Anti-ext} = (c\mathcal{A})^{Anti-int} = c\mathcal{A}$ since, if $\mathcal{A}$ is *A-C* then $c\mathcal{A}$ is *A-O* and $\mathcal{A}^{Anti-int} = \mathcal{A}$ if $\mathcal{A}$ is *A-O*. Hence the result.

(x) Let $x \in \mathcal{A}^{Anti-int} \cap \mathcal{A}^{Anti-ext}$

$\Rightarrow x \in \mathcal{A}^{Anti-int}$ and, $x \in \mathcal{A}^{Anti-ext}$

$\Rightarrow x \in \mathcal{A}^{Anti-int}$ and, $x \in (c\mathcal{A})^{Anti-int}$

$\Rightarrow x \in \mathcal{A}^{Anti-int} \subseteq \mathcal{A}$ and, $x \in \mathcal{A}^{Anti-ext} \subseteq c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}$ and, $x \in c\mathcal{A}$, which is not possible.

Hence, $\mathcal{A}^{Anti-int} \cap \mathcal{A}^{Anti-ext} = \emptyset$.

Definition 4.2.2

For an A-TS $(\mathcal{X}, \mathcal{T})$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, the anti-exterior operator on the space $\mathcal{X}$ is a function: $Anti - ext: \mathcal{I}(\mathcal{X}) \to \mathcal{I}(\mathcal{X})$ such that:

(i) $\mathcal{A}^{Anti-ext} \subseteq c\mathcal{A}$

(ii) $\mathcal{A}^{Anti-ext} = \left[c\left(\mathcal{A}^{Anti-ext}\right)\right]^{Anti-ext}$

(iii) $(\mathcal{A} \cup \mathcal{B})^{Anti-ext} \subseteq \left(\mathcal{A}^{Anti-ext}\right) \cap \left(\mathcal{B}^{Anti-ext}\right)$

(iv) $(\mathcal{A} \cap \mathcal{B})^{Anti-ext} \supseteq \left(\mathcal{A}^{Anti-ext}\right) \cup \left(\mathcal{B}^{Anti-ext}\right)$

4.3 Closure in Anti-Topological Spaces

Definition 4.3.1

Let $(\mathcal{X}, \mathcal{T})$ be an A-TS and $\mathcal{A} \in \mathcal{T}$, then the complement of $\mathcal{A}$, i.e., $c\mathcal{A}$ will be called as anti-closed (A-C).

Proposition 4.3.1

In an ATS $(\mathcal{X}, \mathcal{T})$*, all of (i), (ii) and (iii) will satisfy*:

(i) *The null set and the whole set will not be A-C.*

(ii) *Union of members of* $\mathcal{T}$ *will not be A-C.*

(iii) *Intersection of members of* $\mathcal{T}$ *will not be A-C.*

Definition 4.3.2

For an A-TS $(\mathcal{X}, \mathcal{T})$ *and* $\mathcal{A} \subseteq \mathcal{X}$*, the anti-closure of* $\mathcal{A}$ *will be the intersection of the A-C supersets of* $\mathcal{A}$ *and will be denoted by* $\mathcal{A}^{Anti-cl}$.

Thus, $\mathcal{A}^{Anti-cl} = \cap \{\mathcal{C}_i : \mathcal{A} \subseteq \mathcal{C}_i$ *and each* $\mathcal{C}_i$ *is A-CS*$\}$

We define: $\mathcal{X}^{Anti-cl} = \mathcal{X}$ *and* $\emptyset^{Anti-cl} = \emptyset$.

Remark 4.3.1

In an *A-TS*, since the null set is not *A-O*, so the whole set will not be *A-C* and as such while trying to find the closure of subsets of the whole set, in context, there will be instances that there will be no *A-C* supersets of many subsets. Under such circumstances, we have to conclude that the anti-closures of such sets do not exist. In general, we will assume that the anti-closure exists in order to establish results with respect to the anti-closure. Thus, in all the results that follow in the few propositions below, it has been assumed that the anti-closures exist for the subsets we have considered.

Proposition 4.3.2

For an A-TS $(\mathcal{X}, \mathcal{T})$ *and* $\mathcal{A} \subseteq \mathcal{X}$*, if* $\mathcal{A}$ *is A-C then* $\mathcal{A}^{Anti-cl} = \mathcal{A}$.

Remark 4.3.2

The converse of the above proposition is not always and can be observed from the following example. Let $\mathcal{X} = \{a, b, c, d, e\}$ and $\mathcal{T} = \{\{a\}, \{b\}, \{c, d\}, \{d, e\}\}$, the *A-C* subsets are: $\{b, c, d, e\}, \{a, c, d, e\}, \{a, b, e\}, \{a, b, c\}$. Consider $\mathcal{A} = \{c, d, e\}$, then $\mathcal{A}^{Anti-cl} = \{b, c, d, e\} \cap \{a, c, d, e\} = \{c, d, e\} = \mathcal{A}$. But, $\mathcal{A}$ is not an *A-C* subset of $\mathcal{X}$.

Remark 4.3.3

The anti-closure of a subset $\mathcal{A}$of an *A-TS* $(\mathcal{X}, \mathcal{T})$ is not the largest *A-CS* containing the set $\mathcal{A}$. The counter example provided in ***remark 4.3.2*** illustrates the fact.

Proposition 4.3.3

Let $(\mathcal{X}, \mathcal{T})$ be an ATS and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the following holds:

(i) $\mathcal{A} \subseteq \mathcal{A}^{Anti-cl}$

(ii) $(\mathcal{A}^{Anti-cl})^{Anti-cl} = \mathcal{A}^{Anti-cl}$

(iii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{Anti-cl} \subseteq \mathcal{B}^{Anti-cl}$

(iv) $\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$

(v) $(\mathcal{A} \cap \mathcal{B})^{Anti-cl} \subseteq \mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}$

Proof:

(i) By definition, $\mathcal{A}^{Anti-cl} = \cap \{\mathcal{C}_i : \mathcal{A} \subseteq \mathcal{C}_i$ and each $\mathcal{C}_i$ is *A-CS*$\} \supseteq \mathcal{A}$.

(ii) We have $\mathcal{A}^{Anti-cl} = \cap \{\mathcal{C}_i : \mathcal{A} \subseteq \mathcal{C}_i$and each $\mathcal{C}_i$ is *ACS*$\} = \mathcal{B}$ (say). Here $\mathcal{B}$ is the smallest superset of $\mathcal{A}$. If $\mathcal{B}$ is *A-C*, then $\mathcal{B}^{Anti-cl} = \mathcal{B}$ by ***proposition 4.3.2*** and we have the result. However, if $\mathcal{B}$ is not *A-C*, which is possible by ***remarks 4.3.2*** and ***4.3.3***, then $\mathcal{B}^{Anti-cl} = \cap \{\mathcal{E} : \mathcal{B} \subseteq \mathcal{E}$ and $\mathcal{E}\ is\ A - CS\} = \cap \{\mathcal{F} : \mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{F}$ is $A - CS\} = \mathcal{B}$, because $\mathcal{B}$ is the smallest superset of $\mathcal{A}$ and there will be no other supersets other than those that are larger than $\mathcal{B}$ and all of which are supersets of $\mathcal{A}$.

(iii) By (*i*) we have: $\mathcal{A} \subseteq \mathcal{A}^{Anti-cl}$ and $\mathcal{B} \subseteq \mathcal{B}^{Anti-cl}$.

Now, $\mathcal{B}^{Anti-cl} = \cap \{\mathcal{K} : \mathcal{B} \subseteq \mathcal{K};\ \mathcal{K}\ is\ A - CS\}$ and $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{Nt-cl} = \cap \{\mathcal{L} : \mathcal{A} \subseteq \mathcal{L}, \mathcal{L}\ is\ A - CS\} \subseteq \cap \{\mathcal{K} : \mathcal{B} \subseteq \mathcal{K}\} = \mathcal{B}^{Anti-cl}$ and hence the result.

(iv) By *(iii)*, $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{A}^{Anti-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$

And $\mathcal{B} \subseteq \mathcal{A} \cup \mathcal{B} \Rightarrow \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$

Hence $\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$.

(v) By *(iii)*,$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{Anti-cl} \subseteq \mathcal{A}^{Anti-cl}$

And $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \Rightarrow (\mathcal{A} \cap \mathcal{B})^{Anti-cl} \subseteq \mathcal{B}^{Anti-cl}$

Hence, $(\mathcal{A} \cap \mathcal{B})^{Anti-cl} \subseteq \mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}$.

Remark 4.3.4

Equality will not hold in ***proposition 4.3.3 (iv)***, and can be seen from the example that follows:

Assume that $\mathcal{X} = \{1,2,3,4,5\}$ and $\mathcal{T} = \{\{1\}, \{2,3\}, \{2,4\}, \{3,4\}, \{5\}\}$where the *A-CSs* are: $\{2,3,4,5\}, \{1,4,5\}, \{1,3,5\}, \{1,2,5\}, \{1,2,3,4\}$. Consider $\mathcal{A} = \{1,2\}$ and $\mathcal{B} = \{3\}$, then

$\mathcal{A}^{Anti-cl} = \{1,2\}$ and $\mathcal{B}^{Anti-cl} = \{3\}$ and as such $\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl} = \{1,2,3\}$. Now, $\mathcal{A} \cup \mathcal{B} = \{1,2,3\}$ and as such we have $(\mathcal{A} \cup \mathcal{B})^{Anti-cl} = \{1,2,3,4\}$.

Hence, $\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl} \neq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$.

Definition 4.3.3

If $(\mathcal{X}, \mathcal{T})$ is an ATS, with $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, then the anti-closure operator on the space $\mathcal{X}$ is a function: $Anti - cl$: $\mathcal{I}(\mathcal{X}) \to \mathcal{I}(\mathcal{X})$ such that:

(i) $\mathcal{A} \subseteq \mathcal{A}^{Anti-cl}$

(ii) $\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A} \cup \mathcal{B})^{Anti-cl}$

(iii) $(\mathcal{A}^{Anti-cl})^{Anti-cl} = \mathcal{A}^{Anti-cl}$

Proposition 4.3.4

Let $(\mathcal{X}, \mathcal{T})$ be an A-TS and $\mathcal{A} \subseteq \mathcal{X}$, the we have the following relations between the anti-interior and anti-closure:

(i) $c(\mathcal{A}^{Anti-int}) = (c\mathcal{A})^{Anti-cl}$

(ii) $(c\mathcal{A})^{Anti-int} = c(\mathcal{A}^{Anti-cl})$

(iii) $\mathcal{A}^{Anti-int} = c((c\mathcal{A})^{Anti-cl})$

(iv) $c((c\mathcal{A})^{Anti-int}) = \mathcal{A}^{Anti-cl}$.

(v) $(\mathcal{A} \setminus \mathcal{B})^{Anti-int} = \mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl}$

(vi) $(\mathcal{A} \setminus \mathcal{B})^{Anti-cl} = \mathcal{A}^{Anti-cl} \setminus \mathcal{B}^{Anti-int}$

Proof:

(i) We have: $\mathcal{A}^{Anti-int} = \cup\, \mathcal{O}_i$ so that each $\mathcal{O}_i$ is *A-OS* and $\mathcal{O}_i \subseteq \mathcal{A}$.

Thus, $c(\mathcal{A}^{Anti-int}) = c(\cup\, \mathcal{O}_i)$ so that $c(\mathcal{O}_i) \supseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{Anti-int}) = \cap\, (c\mathcal{O}_i)$ so that each $c\mathcal{O}_i$is *A-CS and* $c(\mathcal{A}) \subseteq c(\mathcal{O}_i)$

Or, $c(\mathcal{A}^{Anti-int}) = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ *is A-CS* and $c(\mathcal{A}) \subseteq \mathcal{C}_i$

Or, $c(\mathcal{A}^{Anti-int}) = (c\mathcal{A})^{Anti-cl}$

(ii) We have: $\mathcal{A}^{Anti-cl} = \cap\, \mathcal{C}_i$ so that each $\mathcal{C}_i$ is *A-CS* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Thus, $c(\mathcal{A}^{Anti-cl}) = c(\cap\, \mathcal{C}_i)$ so that $c(\mathcal{A}) \supseteq c(\mathcal{C}_i)$

Or, $c(\mathcal{A}^{Anti-cl}) = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *A-OS* and $c(\mathcal{C}_i) \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{Anti-cl}) = \cup\, (\mathcal{O}_i)$ so that each $\mathcal{O}_i$ is *A-OS* and $\cup\, (\mathcal{O}_i) \subseteq c(\mathcal{A})$

Or, $c(\mathcal{A}^{Anti-cl}) = (c\mathcal{A})^{Anti-int}$

(iii) We have$(c\mathcal{A})^{Anti-cl} = \cap\, \mathcal{C}_i$, where each $\mathcal{C}_i$ is *A-CS* and $c\mathcal{A} \subseteq \mathcal{C}_i$.

So, $c(c\mathcal{A})^{Anti-cl} = c(\cap\, \mathcal{C}_i)$ so that $c(c\mathcal{A}) \supseteq c\mathcal{C}_i$

Or, $c(c\mathcal{A})^{Anti-cl} = \cup\, (c\mathcal{C}_i)$ so that each $c\mathcal{C}_i$ is *A-OS* and $c\mathcal{C}_i \subseteq \mathcal{A}$

Or, $c(c\mathcal{A})^{Anti-cl} = \cup\, (\mathcal{D}_i)$ so that each $\mathcal{D}_i$ is *A-OS* and $\mathcal{D}_i \subseteq \mathcal{A}$.

Or, $c(c\mathcal{A})^{Anti-cl} = \mathcal{A}^{Anti-int}$.

(iv) We have$(c\mathcal{A})^{Anti-int} = \cup\, \mathcal{B}_i$ so that each $\mathcal{B}_i$ is *A-OS* and $\mathcal{B}_i \subseteq c\mathcal{A}$

So, $c((c\mathcal{A})^{Anti-int}) = c(\cup\, \mathcal{B}_i)$ so that each $\mathcal{B}_i$ is *A-OS* and $\mathcal{B}_i \subseteq c\mathcal{A}$

Or, $c((c\mathcal{A})^{Anti-int}) = \cap\, (c\mathcal{B}_i)$ so that each $c\mathcal{B}_i$ is *A-CS* and $c\mathcal{B}_i \supseteq c(c\mathcal{A})$

Or, $c((c\mathcal{A})^{Anti-int}) = \cap\, (\mathcal{C}_i)$ so that each $\mathcal{C}_i$ is *A-CS* and $\mathcal{A} \subseteq \mathcal{C}_i$.

Or, $c((c\mathcal{A})^{Anti-int}) = \mathcal{A}^{Anti-cl}$.

(v) We have:

$$
\begin{aligned}
\mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl} &= \mathcal{A}^{Anti-int} \cap c(\mathcal{B}^{Anti-cl}) \\
&= \mathcal{A}^{Anti-int} \cap (c\mathcal{B})^{Anti-int} \text{ by (ii).} \\
&\supseteq (\mathcal{A} \cap c\mathcal{B})^{Anti-int} \text{ by \textbf{proposition 4.1.2 (iv).}} \\
&= (\mathcal{A} \setminus \mathcal{B})^{Anti-int}
\end{aligned}
$$

Hence, $(\mathcal{A} \setminus \mathcal{B})^{Anti-int} \subseteq \mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl}$

Conversely, let $x \in \mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl}$

$\Rightarrow x \in \mathcal{A}^{Anti-int}$ but $x \notin \mathcal{B}^{Anti-cl}$

$\Rightarrow x \in \mathcal{A}$ but $x \notin \mathcal{B}$

$\Rightarrow x \in \mathcal{A} \setminus \mathcal{B}$

$\Rightarrow x \in (\mathcal{A} \setminus \mathcal{B})^{Anti-int}$ as $x \in \mathcal{A}^{Anti-int}$

Thus, $\mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A} \setminus \mathcal{B})^{Anti-int}$

Hence, $(\mathcal{A} \setminus \mathcal{B})^{Anti-int} = \mathcal{A}^{Anti-int} \setminus \mathcal{B}^{Anti-cl}$

(vi) We have

$$
\begin{aligned}
(\mathcal{A} \setminus \mathcal{B})^{Anti-cl} &= (\mathcal{A} \cap c\mathcal{B})^{Anti-cl} \\
&\subseteq \mathcal{A}^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl} \\
&= \mathcal{A}^{Anti-cl} \cap c(\mathcal{B}^{Anti-int}) \text{ by } (i) \\
&= \mathcal{A}^{Anti-cl} \setminus \mathcal{B}^{Anti-int}
\end{aligned}
$$

Thus, $(\mathcal{A} \setminus \mathcal{B})^{Anti-cl} \subseteq \mathcal{A}^{Anti-cl} \setminus \mathcal{B}^{Anti-int}$.

Conversely, let $x \in \mathcal{A}^{Anti-cl} \setminus \mathcal{B}^{Anti-int}$

$\Rightarrow x \in \mathcal{A}^{Anti-cl}$ but $x \notin \mathcal{B}^{Anti-int}$

$\Rightarrow x \in \mathcal{A}^{Anti-cl}$ but $x \notin \mathcal{B}$.

$$\Rightarrow x \in (\mathcal{A} \setminus \mathcal{B})^{Anti-cl} \text{ since } x \in \mathcal{A}^{Anti-cl}$$

Hence, $\mathcal{A}^{Anti-cl} \setminus \mathcal{B}^{Anti-int} \subseteq (\mathcal{A} \setminus \mathcal{B})^{Anti-cl}$.

Definition 4.3.4

A proper subset $\mathcal{A}$ of an A-TS $(\mathcal{X}, \mathcal{T})$ is termed anti-clopen set if it is both an anti-open and an anti-closed set.

Remark 4.3.5

Remark 2.3.6 of ***chapter 2*** states that a *N-T* cannot be a neutro-clopen topology because the whole set and the null set are not present in a *N-T* simultaneously. If one of the two is present, the other cannot be present and because of this, since they are complements of each other, either the whole set or the empty set will not be a neutro-clopen set even if the other subsets of the *N-T* are all neutro-clopen. However, in the *A-T*, since both the whole set and the null set are not *A-O*, so an *A-T* can be an anti-clopen topology. For example, we may take assume $\mathcal{X} = \{1,2,3,4\}$ and consider the *A-T* given by $\mathcal{T} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ which is obviously an anti-clopen topology.

4.4 Boundary in Anti-Topological Spaces

Definition 4.4.1

For an A-TS $(\mathcal{X}, \mathcal{T})$ and $\mathcal{A} \subseteq \mathcal{X}$, the anti-boundary of $\mathcal{A}$, denoted by $\mathcal{A}^{Anti-bd}$, is defined as $\mathcal{A}^{Anti-bd} = \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}$. In other words, the anti-boundary of $\mathcal{A}$ consist of all those points that belong to the anti-closure of $\mathcal{A}$ and the anti-closure of the complement of $\mathcal{A}$.

Remark 4.4.1

The points that belong to the anti-boundary of $\mathcal{A}$ will be the points that will be neither included in the anti-interior of $\mathcal{A}$ nor the anti-exterior of $\mathcal{A}$. An example may be considered to have a clearer glimpse to the context. Let $\mathcal{X} = \{1,2,3,4\}$, $\mathcal{T} = \{\{1\},\{2,3\},\{2,4\},\{3,4\}\}$ and $\mathcal{A} = \{1,2\}$ then $\mathcal{A}^{Anti-int} = \{1\}$, $\mathcal{A}^{Anti-ext} = \{3,4\}$ and $\mathcal{A}^{Anti-bd} = \{2\}$, as $\mathcal{A}^{Anti-cl} = \{1,2\}$ and $(c\mathcal{A})^{Anti-cl} = \{2,3,4\}$.

Proposition 4.4.1

If $\mathcal{A}$ is any proper subset of an ATS $(\mathcal{X}, \mathcal{T})$ then the following results are true:

(i) $\mathcal{A}^{Anti-bd} = c(\mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext})$

(ii) $\mathcal{X} = \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext} \cup \mathcal{A}^{Anti-bd}$

(iii) $\mathcal{A}^{Anti-bd} = \mathcal{A}^{Anti-cl} \setminus \mathcal{A}^{Anti-int}$

(iv) $\mathcal{A}^{Anti-int} \cup (c\mathcal{A})^{Anti-int} = c(\mathcal{A}^{Anti-bd})$

(v) $\mathcal{A}^{Anti-int} = \mathcal{A} \setminus \mathcal{A}^{Anti-bd}$

(vi) $\mathcal{A}^{Anti-cl} = \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-bd}$.

Proof:

(i) By definition, if $x \in \mathcal{A}^{Anti-bd}$ then $x \notin \mathcal{A}^{Anti-int}$ and $x \notin \mathcal{A}^{Anti-ext}$

$$\Leftrightarrow x \notin \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext}$$

$$\Leftrightarrow x \in c(\mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext})$$

Hence, $\mathcal{A}^{Anti-bd} = c(\mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext})$

(ii) From *(i)* we have*:* $\mathcal{A}^{Anti-bd} = c(\mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext})$which leads to the results: $\mathcal{A}^{Anti-bd} \cap \mathcal{A}^{Anti-int} = \emptyset$ and $\mathcal{A}^{Anti-bd} \cap \mathcal{A}^{Anti-ext} = \emptyset$ thereby leading to the conclusion that*:* $\mathcal{X} = \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext} \cup \mathcal{A}^{Anti-bd}$

(iii) From *(i)*, we have:

$$\mathcal{A}^{Anti-bd} = \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}$$

$$= \mathcal{A}^{Anti-cl} \setminus c((c\mathcal{A})^{Anti-cl})$$

$$= \mathcal{A}^{Anti-cl} \setminus \mathcal{A}^{Anti-int},$$ [by ***proposition 4.3.4 (iii)***]

(iv) We have: $c\left(\mathcal{A}^{Anti-bd}\right) = c(\mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl})$

$$\Rightarrow c\left(\mathcal{A}^{Anti-bd}\right) = c(\mathcal{A}^{Anti-cl}) \cup c((c\mathcal{A})^{Anti-cl})$$

$$\Rightarrow c\left(\mathcal{A}^{Anti-bd}\right) = (c\mathcal{A})^{Anti-int} \cup \mathcal{A}^{Anti-int},$$

[by ***proposition 4.3.4 (ii)*** and ***(iii)***]

Hence, $\mathcal{A}^{Anti-int} \cup (c\mathcal{A})^{Anti-int} = c(\mathcal{A}^{Anti-bd})$.

(v) Let $x \in \mathcal{A}^{Anti-int}$

So, $x \in \mathcal{A}$ but $x \notin c\mathcal{A}$

$\Rightarrow x \in \mathcal{A}$ and $x \in \mathcal{A}^{Anti-cl}$ but $x \notin c\mathcal{A}^{Anti-cl}$

$\Rightarrow x \in \mathcal{A}$ and ($x \in \mathcal{A}^{Anti-cl}$ but $x \notin c\mathcal{A}^{Anti-cl}$)

$\Rightarrow x \in \mathcal{A}$ but $x \notin \mathcal{A}^{Anti-bd}$

$\Rightarrow x \in \mathcal{A} \setminus \mathcal{A}^{Anti-bd}$

Hence, $\mathcal{A}^{Anti-int} \subseteq \mathcal{A} \setminus \mathcal{A}^{Anti-bd}$.

Conversely, let $x \in \mathcal{A} \setminus \mathcal{A}^{Anti-bd}$.

Then $x \in \mathcal{A}$ but $x \notin \mathcal{A}^{Anti-bd}$, so there will be an *A-OS* $\mathcal{O}_x$ that contain x such that $\mathcal{O}_x \cap c\mathcal{A} = \emptyset$ and $x \in \mathcal{O}_x \subseteq \mathcal{A}$ which shows that $x \in \mathcal{A}^{Anti-int}$.

Thus, $\mathcal{A} \setminus \mathcal{A}^{Anti-bd} \subseteq \mathcal{A}^{Anti-int}$.

Hence, we have: $\mathcal{A}^{Anti-int} = \mathcal{A} \setminus \mathcal{A}^{Anti-bd}$

(vi) We have $\mathcal{A}^{Anti-cl} = \cap \{\mathcal{C}: \mathcal{C}$ *AC* with $\mathcal{A} \subseteq \mathcal{C}\}$

Hence, $c(\mathcal{A}^{Anti-cl}) = c[\cap \{\mathcal{C}: \mathcal{C}$ is *ACS* with $\mathcal{A} \subseteq \mathcal{C}\}]$

$= \cup \{c\mathcal{C}: c\mathcal{C}$ is *AOS* with $c\mathcal{C} \subseteq c\mathcal{A}\}$

$= \mathcal{A}^{Anti-ext}$

Hence, $c\{c(\mathcal{A}^{Anti-cl})\} = c(\mathcal{A}^{Anti-ext}) = \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-bd}$, by *(ii)*.

Thus, $\mathcal{A}^{Anti-cl} = \mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-bd}$.

Proposition 4.4.2

If $\mathcal{A}$,$\mathcal{B}$ are arbitrary subsets of an A-TS $\mathcal{X}$ then:

(i) $\emptyset^{Anti-bd} = \emptyset$

(ii) $\mathcal{A}^{Anti-bd} = (c\mathcal{A})^{Anti-bd}$

Proof*:*

(i) By ***proposition 4.4.1 (i)***, we have: $\mathcal{A}^{Anti-bd} = c(\mathcal{A}^{Anti-int} \cup \mathcal{A}^{Anti-ext})$, wherein replacing $\mathcal{A}$ by $\emptyset$, we get: $\emptyset^{Anti-bd} = c\left(\emptyset^{Anti-int} \cup \emptyset^{Anti-ext}\right) = c(\emptyset \cup \mathcal{X}) = c(\mathcal{X}) = \emptyset$.

(ii) We have: $(c\mathcal{A})^{Anti-bd} = (c\mathcal{A})^{Anti-cl} \cap \{c(c\mathcal{A})\}^{Anti-cl} = (c\mathcal{A})^{Anti-cl} \cap \mathcal{A}^{Anti-cl} = \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl} = \mathcal{A}^{Anti-bd}$

Remark 4.4.2

In an *A-TS* $\mathcal{X}$, the boundary of $\mathcal{A} \subseteq \mathcal{X}$ is not necessarily *A-CS* and can be seen from the example that follows:

We may take $\mathcal{X} = \{k_1, k_2, k_3, k_4, k_5\}$ with $\mathcal{T} = \{\{k_1\}, \{k_2\}, \{k_3, k_4\}, \{k_3, k_5\}, \{k_4, k_5\}\}$, then clearly $(\mathcal{X}, \mathcal{T})$is an *ATS*. Let$\mathcal{A} = \{k_1, k_2, k_3\}$, then $c\mathcal{A} = \{k_4, k_5\}$ and $\mathcal{A}^{Anti-cl} = \{k_1, k_2, k_3\}$ and $(c\mathcal{A})^{Anti-cl} = \{k_3, k_4, k_5\}$ and so $\mathcal{A}^{Anti-bd} = \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl} = \{k_3\}$ which is not *A-C*. However, in a *GTS*, the boundary of a subset of a *GTS* is a closed set which has however been seen to be not true in the case of a subset of an *A-TS.*

Proposition 4.4.3

If $\mathcal{A},\mathcal{B}$ are arbitrary proper subsets of an A-TS $\mathcal{X}$ then the following are true:

(i) $(\mathcal{A}^{Anti-int})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd}$

(ii) $(\mathcal{A}^{Anti-cl})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd}$

(iii) $(\mathcal{A} \cap \mathcal{B})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$

(iv) $(\mathcal{A} \cup \mathcal{B})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$

Proof:

(i) On applying the definition of the anti-boundary, we have:

$(\mathcal{A}^{Anti-int})^{Anti-bd} = (\mathcal{A}^{Anti-int})^{Anti-cl} \cap [\{c(\mathcal{A}^{Anti-int})\}^{Anti-cl}]$.

$= (\mathcal{A}^{Anti-int})^{Anti-cl} \cap [((c\mathcal{A})^{Anti-cl})^{Anti-cl}]$, [by ***proposition 4.3.4 (i)***]

$= (\mathcal{A}^{Anti-int})^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}$, [by ***proposition 4.3.3 (ii)***]

$\subseteq \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}$, since $\mathcal{A}^{Anti-int} \subseteq \mathcal{A}$

$= \mathcal{A}^{Anti-bd}$

Hence $(\mathcal{A}^{Anti-int})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd}$.

(ii) We have,$(\mathcal{A}^{Anti-cl})^{Anti-bd} = (\mathcal{A}^{Anti-cl})^{Anti-cl} \cap (c(\mathcal{A}^{Anti-cl}))^{Anti-cl}$

$= \mathcal{A}^{Anti-cl} \cap (c(\mathcal{A}^{Anti-cl}))^{Anti-cl}$ [by ***proposition 4.3.3 (ii)***]

Now, $\mathcal{A} \subseteq \mathcal{A}^{Anti-cl} \Rightarrow c(\mathcal{A}^{Anti-cl}) \subseteq c\mathcal{A}$

$\Rightarrow (c(\mathcal{A}^{Anti-cl}))^{Anti-cl} \subseteq (c\mathcal{A})^{Anti-cl}$, [by ***proposition 4.3.3 (iii)***]

Thus, $(\mathcal{A}^{Anti-cl})^{Anti-bd} = \mathcal{A}^{Anti-cl} \cap (c(\mathcal{A}^{Anti-cl}))^{Anti-cl}$

$\subseteq \mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl} = \mathcal{A}^{Anti-bd}$

Hence, $(\mathcal{A}^{Anti-cl})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd}$

(iii) On applying the definition, we get:

$(\mathcal{A} \cap \mathcal{B})^{Anti-bd} = (\mathcal{A} \cap \mathcal{B})^{Anti-cl} \cap (c(\mathcal{A} \cap \mathcal{B}))^{Anti-cl}$

$\subseteq [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}] \cap [(c\mathcal{A} \cup c\mathcal{B})^{Anti-cl}]$

$\subseteq [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}] \cap [(c\mathcal{A})^{Anti-cl} \cup (c\mathcal{B})^{Anti-cl}]$

$= [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}] \cup [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}$
$\cap (c\mathcal{B})^{Anti-cl}]$

$= [\mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl} \cap \mathcal{B}^{Anti-cl}] \cup [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl}$
$\cap (c\mathcal{B})^{Anti-cl}]$

$= [\{\mathcal{A}^{Anti-cl} \cap (c\mathcal{A})^{Anti-cl}\} \cap \mathcal{B}^{Anti-cl}] \cup [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl}]$

$= [(\mathcal{A})^{Anti-bd} \cap \mathcal{B}^{Anti-cl}] \cup [\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-bd}]$

$\subseteq (\mathcal{A})^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$, since $(\mathcal{A})^{Anti-bd} \cap \mathcal{B}^{Anti-cl} \subseteq (\mathcal{A})^{Anti-bd}$ and also, $\mathcal{A}^{Anti-cl} \cap \mathcal{B}^{Anti-bd} \subseteq (\mathcal{B})^{Anti-bd}$

Hence, $(\mathcal{A} \cap \mathcal{B})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$.

(iv) On applying the definition, we get:

$(\mathcal{A} \cup \mathcal{B})^{Anti-bd} = (\mathcal{A} \cup \mathcal{B})^{Anti-cl} \cap (c(\mathcal{A} \cup \mathcal{B}))^{Anti-cl}$

$\subseteq [\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl}] \cap [(c\mathcal{A})^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl}]$

$= [(c\mathcal{A})^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl}] \cap [\mathcal{A}^{Anti-cl} \cup \mathcal{B}^{Anti-cl}]$

$= [(c\mathcal{A})^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl} \cap \mathcal{A}^{Anti-cl}] \cup [(c\mathcal{A})^{Anti-cl} \cap (c\mathcal{B})^{Anti-cl} \cap \mathcal{B}^{Anti-cl}]$

$= [\{(c\mathcal{A})^{Anti-cl} \cap \mathcal{A}^{Anti-cl}\} \cap (c\mathcal{B})^{Anti-cl}] \cup [(c\mathcal{A})^{Anti-cl} \cap \{(c\mathcal{B})^{Anti-cl} \cap \mathcal{B}^{Anti-cl}\}]$

$= [\mathcal{A}^{Anti-bd} \cap (c\mathcal{B})^{Anti-cl}] \cup [(c\mathcal{A})^{Anti-cl} \cap \mathcal{B}^{Anti-bd}]$

$\subseteq \mathcal{A}^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$, since $\mathcal{A}^{Anti-bd} \cap (c\mathcal{B})^{Anti-cl} \subseteq \mathcal{A}^{Anti-bd}$ and $(c\mathcal{A})^{Anti-cl} \cap \mathcal{B}^{Anti-bd} \subseteq \mathcal{B}^{Anti-bd}$.

Hence, $(\mathcal{A} \cup \mathcal{B})^{Anti-bd} \subseteq \mathcal{A}^{Anti-bd} \cup \mathcal{B}^{Anti-bd}$.

Remark 4.4.3

That the equality does not hold in ***(i), (ii)*** of ***proposition 4.4.3***, can be illustrated as follows:

(i) For the inequality in ***proposition 4.4.3 (i)***, let us consider $\mathcal{X} = \{l_1, l_2, l_3, l_4, l_5\}$ *and* $\mathcal{T} = \{\{l_1\}, \{l_2\}, \{l_3, l_4\}, \{l_3, l_5\}, \{l_4, l_5\}\}$, clearly $(\mathcal{X}, \mathcal{T})$ is an *A-TS.* Here the *A-CS* are: $\{l_2, l_3, l_4, l_5\}$, $\{l_1, l_3, l_4, l_5\}, \{l_1, l_2, l_5\}, \{l_1, l_2, l_4\}, \{l_1, l_2, l_3\}$. *Let* $\mathcal{A} = \{l_1, l_2, l_3\}$, *then* $c\mathcal{A} = \{l_4, l_5\}$, $\mathcal{A}^{Anti-int} = \{l_1, l_2\}$, $\mathcal{A}^{Anti-cl} = \{l_1, l_2, l_3\}$, $(c\mathcal{A})^{Anti-cl} = \{l_3, l_4, l_5\}$ and so $\mathcal{A}^{Anti-bd} = \{l_1, l_2, l_3\} \cap \{l_3, l_4, l_5\} = \{l_3\}$. *Also,* $(\mathcal{A}^{Anti-int})^{Anti-cl} = \{l_1, l_2\}^{Anti-cl} = \{l_1, l_2\}$ and $(c(\mathcal{A}^{Anti-int}))^{Anti-cl} = \{l_3, l_4, l_5\}^{Anti-cl} = \{l_3, l_4, l_5\}$. So, $(\mathcal{A}^{Anti-int})^{Anti-bd} = \emptyset$. Thus, $\emptyset = (\mathcal{A}^{Anti-int})^{Anti-bd} \neq \mathcal{A}^{Anti-bd} = \{l_3\}$

(ii) For the inequality in ***proposition 4.4.3 (ii)***, let us take $\mathcal{X} = \{l_1, l_2, l_3, l_4, l_5\}$ and $\mathcal{T} = \{\{l_1\}, \{l_4\}, \{l_2, l_3\}, \{l_2, l_5\}, \{l_3, l_5\}\}$, clearly $(\mathcal{X}, \mathcal{T})$ is an *A-TS.* Here the *A-CS* are*:* $\{l_2, l_3, l_4, l_5\}, \{l_1, l_2, l_3, l_5\}, \{l_1, l_4, l_5\}, \{l_1, l_3, l_4\}, \{l_1, l_2, l_4\}$. Let $\mathcal{A} = \{l_1, l_2, l_3\}$, then $c\mathcal{A} = \{l_4, l_5\}$, and $\mathcal{A}^{Anti-cl} = \{l_1, l_2, l_3, l_5\}$, $(c\mathcal{A})^{Anti-cl} = \{l_4, l_5\}$, $\mathcal{A}^{Anti-bd} = \{l_5\}$.

Now, $(\mathcal{A}^{Anti-cl})^{Anti-cl} = \{l_1, l_2, l_3, l_5\}$ and $(c(\mathcal{A}^{Anti-cl}))^{Anti-cl} = (\{l_4\})^{Anti-cl} = \{l_4\}$ and *so,* $(\mathcal{A}^{Anti-cl})^{Anti-bd} = (\mathcal{A}^{Anti-cl})^{Anti-cl} \cap (c(\mathcal{A}^{Anti-cl}))^{Anti-cl} = \{l_1, l_2, l_3, l_5\} \cap \{l_4\} = \emptyset$.

Thus, we have: $\emptyset = (\mathcal{A}^{Anti-cl})^{Anti-bd} \neq \mathcal{A}^{Anti-bd} = \{l_5\}$.

4.5 Relative Topology of an Anti-Topological Space

Definition 4.5.1

For an A-TS $(\mathcal{X}, \mathcal{T})$ *and* $\mathcal{A} \subseteq \mathcal{X}$*, we define the relative anti-topology* $\mathcal{T}_{\mathcal{A}}$ *for* $\mathcal{A}$ *to be the collection given by:* $\mathcal{T}_{\mathcal{A}} = \{\mathcal{B} \cap \mathcal{A} : \mathcal{B} \in \mathcal{T}\}$*. The A-TS* $(\mathcal{A}, \mathcal{T}_{\mathcal{A}})$ *is called a sub-space of the A-TS* $(\mathcal{X}, \mathcal{T})$ *and the A-T* $\mathcal{T}_{\mathcal{A}}$ *is said to be induced by* $\mathcal{T}$*.*

Example 4.5.1

Suppose $\mathcal{X} = \{1,2,3,4,5\}$*, then* $\mathcal{T} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}$ *is an A-T on* $\mathcal{X}$*. If* $\mathcal{A} = \{1,3,4\}$ *then* $\mathcal{T}_{\mathcal{A}} = \{\{1,3\}, \{1,4\}, \{3,4\}\}$ *is the relative A-T.*

Proposition 4.5.1

Suppose $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ *is a sub-space of an A-TS* $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$*, then the results that follow are true:*

(i) $\mathcal{A} \subseteq \mathcal{Y}$ *is A-C in* $\mathcal{Y}$ *if and only if there is an A-CS* $\mathcal{C}$ *in* $\mathcal{X}$ *so that* $\mathcal{A} = \mathcal{C} \cap \mathcal{Y}$*.*

(ii) *For every* $\mathcal{A} \subseteq \mathcal{Y}$*,* $\mathcal{A}_{\mathcal{Y}}^{Anti-cl} = \mathcal{A}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{Y}$*, where* $\mathcal{A}_{\mathcal{X}}^{Anti-cl}$ *is the Anti-closure of* $\mathcal{A}$ *in* $\mathcal{X}$*.*

(iii) *A subset* $\mathcal{P}$ *of* $\mathcal{Y}$ *will be a* $\mathcal{T}_{\mathcal{Y}}$*-anti-nhd of a point* $y \in \mathcal{Y}$ *iff* $\mathcal{P} = \mathcal{Q} \cap \mathcal{Y}$ *for some* $\mathcal{T}_{\mathcal{X}}$*-anti-nhd* $\mathcal{Q}$ *of* y*.*

(iv) *For every* $\mathcal{A} \subseteq \mathcal{Y}$*,* $\mathcal{A}_{\mathcal{X}}^{Anti-int} \subseteq \mathcal{A}_{\mathcal{Y}}^{Anti-int}$

Proof:

(i) Let $\mathcal{A}$ be *A-C* in $\mathcal{Y}$

$\Leftrightarrow c\mathcal{A}$ is *A-O* in $\mathcal{Y}$

$\Leftrightarrow c\mathcal{A} = \mathcal{B} \cap \mathcal{Y}$, $\mathcal{B}$ is *A-O* in $\mathcal{X}$

$\Leftrightarrow \mathcal{A} = c(\mathcal{B} \cap \mathcal{Y})$

$\Leftrightarrow \mathcal{A} = c(\mathcal{B}) \cup c(\mathcal{Y})$, De-Morgan's law

$\Leftrightarrow \mathcal{A} = c(\mathcal{B}) \cup \emptyset$, since $c(\mathcal{Y}) = \emptyset \Leftrightarrow \mathcal{A} = c(\mathcal{B}) = \mathcal{Y} \setminus \mathcal{B}$

$\Leftrightarrow \mathcal{A} = \mathcal{Y} \cap c\mathcal{B}$

$\Leftrightarrow \mathcal{A} = \mathcal{Y} \cap \mathcal{C}$, where $\mathcal{C} = c\mathcal{B}$ is *A-C* in $\mathcal{X}$.

(ii) By definition: $\mathcal{A}_{\mathcal{Y}}^{Anti-cl} = \cap \{\mathcal{D}: \mathcal{D}$ is *A-CS* in $\mathcal{Y}$ and $\mathcal{A} \subseteq \mathcal{D}\}$

$= \cap \{\mathcal{C} \cap \mathcal{Y}: \mathcal{C}$ is *A-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C} \cap \mathcal{Y}\}$, by (i)

$= \cap \{\mathcal{C} \cap \mathcal{Y}: \mathcal{C}$ is *A-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C}\}$

$= [\cap \{\mathcal{C}: \mathcal{C}$ is *A-CS* in $\mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{C}\}] \cap \mathcal{Y}$

$= \mathcal{A}_{\mathcal{X}}^{Anti-cl} \cap \mathcal{Y}$, where $\mathcal{A}_{\mathcal{X}}^{Anti-cl}$ is the anti-closure of $\mathcal{A}$ in $\mathcal{X}$.

(iii) Let us assume $\mathcal{P}$ to be a $\mathcal{T}_{\mathcal{Y}}$-anti-nhd of a point y in $\mathcal{Y}$. Then a $\mathcal{T}_{\mathcal{Y}}$-*A-OS* set $\mathcal{K}$ will be there so that $y \in \mathcal{K} \subseteq \mathcal{P}$. Thus, for a $\mathcal{T}_{\mathcal{X}}$-*A-OS* $\mathcal{J}$ we have: $y \in \mathcal{K} = \mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{P}$. Now, if we assume $\mathcal{Q} = \mathcal{P} \cup \mathcal{J}$, then $\mathcal{Q}$ is a $\mathcal{T}_{\mathcal{Y}}$-anti-nhd of y since $\mathcal{J}$ is a $\mathcal{T}_{\mathcal{X}}$-*AOS* such that y $\in \mathcal{J} \subseteq \mathcal{Q}$.

Further, $\mathcal{Q} \cap \mathcal{Y} = (\mathcal{P} \cup \mathcal{J}) \cap \mathcal{Y} = (\mathcal{P} \cap \mathcal{Y}) \cup (\mathcal{J} \cap \mathcal{Y}) = \mathcal{P} \cup (\mathcal{J} \cap \mathcal{Y}) = \mathcal{P}$, since $\mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{P}$.

Conversely, if $\mathcal{P} = \mathcal{Q} \cap \mathcal{Y}$ for some $\mathcal{T}_{\mathcal{Y}}$-anti-nhd $\mathcal{Q}$ of y. Then there exists a $\mathcal{J} \in \mathcal{T}_{\mathcal{X}}$ so that $y \in \mathcal{J} \subseteq \mathcal{Q}$ which means $y \in \mathcal{J} \cap \mathcal{Y} \subseteq \mathcal{Q} \cap \mathcal{Y} = \mathcal{P}$. And since $\mathcal{J} \cap \mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}$, so $\mathcal{P}$is a $\mathcal{T}_{\mathcal{Y}}$-anti-nhd of the point y.

(iv) We have $x \in \mathcal{A}_{\mathcal{X}}^{Anti-int} \Rightarrow x$ is a $\mathcal{T}_{\mathcal{X}}$-interior point of $\mathcal{A} \Rightarrow \mathcal{A}$ is a $\mathcal{T}_{\mathcal{X}}$-anti-nhd of $x \Rightarrow \mathcal{A} \cap \mathcal{Y}$ is a $\mathcal{T}_{\mathcal{Y}}$-anti-nhd of $x \Rightarrow \mathcal{A} \subseteq \mathcal{Y} \Rightarrow x \in \mathcal{A}_{\mathcal{Y}}^{Anti-int}$ and hence we must have $\mathcal{A}_{\mathcal{X}}^{Anti-int} \subseteq \mathcal{A}_{\mathcal{Y}}^{Anti-int}$