CHAPTER 5

Continuity of Functions in Neutro-Topological Spaces and Anti-Topological Spaces

In the current chapter the aspect of continuity of functions is introduced in neutro-topological space (N-TS) with the help of neutro-neighborhoods (Nu-nhd) and N-OSs and continuity properties are analyzed in different types of functions. Further, taking advantage of the fact that a N-TS can be obtained from every GTS, the concept of weakly neutro-continuity is introduced and some of the properties of such a form of continuity are also analyzed. Further, neutro-homeomorphism is also introduced with the help of weakly neutro-continuity of function and some classical properties are analyzed. The concept of continuity of functions in A-TSs has been defined via A-OS. Moreover, the concept of weak continuity could not be extended to the study in A-TSs.

5.1 Continuity in Neutro-Topological Spaces

Definition 5.1.1

For two N-TSs $(\mathcal{X},\mathcal{T}_1)$, $(\mathcal{Y},\mathcal{T}_2)$, *a map* f *defined between* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *will be Nu-continuous at a member* x *of* $\mathcal{X}$ *iff for all* $\mathcal{T}_2$*-Nu-nhd* $\mathcal{Q}$ *of* $f(x)$ *there is a* $\mathcal{T}_1$*-Nu-nhd* $\mathcal{O}$ *of the member* x *so that* $f(\mathcal{O}) \subset \mathcal{Q}$.

Proposition 5.1.1

For two N-TSs $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *a mapping* f *defined between* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *will be Nu-continuous iff for each* $\mathcal{O} \in \mathcal{T}_2$, $f^{-1}(\mathcal{O}) \in \mathcal{T}_1$..

Definition 5.1.2

For two GTSs $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *the structures* $(\mathcal{X},\mathcal{T}_1 \setminus \psi)$ *and* $(\mathcal{Y},\mathcal{T}_2 \setminus \psi)$ *where* ψ *may be* $\emptyset$ *or* $\mathcal{X}$, *are N-TSs. A function which is continuous with respect to these N-Ts will be called weakly Nu-continuous.*

Some of the results discussed in this chapter have been published in: Basumatary B., & Khaklary J.K. (2024). A Study on Continuity functions in neutro-topological spaces. *Neutrosophic Sets and Systems*, **78**, 341-352.

Remark 5.1.1

We denote the topologies $\mathcal{T}_1 \setminus \psi$ and $\mathcal{T}_2 \setminus \psi$ with the symbol $\mathcal{C}$ wherever necessary. That is, $\mathcal{C}$ will denote $\mathcal{T}_1 \setminus \psi$ or $\mathcal{T}_2 \setminus \psi$ with $\psi = \emptyset$, or $\mathcal{X}$. It may be observed that in the *N-TS* $(\mathcal{X}, \mathcal{C})$, the union or the intersection of *N-OS* are *N-O*. If a function is weakly Nu-continuous then the properties of union or the intersection of the *N-OS* in the *N-TS* is preserved in the resulting *N-T* from the parent topology from which the whole set or the null set is excluded. Moreover, once a function is termed weakly Nu-continuous, properties of closure and interior will also be preserved. While dealing with closure properties, it may be assumed that the *N-T* that is in use is $\mathcal{T}_1 \setminus \emptyset$ and $\mathcal{T}_2 \setminus \emptyset$ and while dealing with interior properties, it might be assumed that the *N-Ts* $\mathcal{T}_1 \setminus \mathcal{X}$ and $\mathcal{T}_2 \setminus \mathcal{X}$ are in use.

Proposition 5.1.2

If a map is Nu-continuous then it is also weakly Nu-continuous.

Proof:

If ζ, a map between $\mathcal{T}_1$ and $\mathcal{T}_2$ is Nu-continuous, then if $\mathcal{W} \in \mathcal{T}_2$ then $f^{-1}(\mathcal{W}) \in \mathcal{T}_1$. That is, if $\mathcal{W}$ is $\mathcal{T}_2$-*N-O* then $f^{-1}(\mathcal{W})$ is $\mathcal{T}_1$-*N-O*. Since in a *N-TS*, the null set or the whole set do not simultaneously belong to the *N-T* and also in the *N-TS* $(\mathcal{X}, \mathcal{C})$, the null set or the whole set are excluded and as such every $\mathcal{T}_2$-*N-OS* will be $\mathcal{C}$-*N-O* and every $\mathcal{T}_1$-*N-OS* will be $\mathcal{C}$-*N-O* and thus the map f between $(\mathcal{X}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{C})$ will be Nu-continuous. Thus f is weakly Nu-continuous.

Remark 5.1.2

Proposition 5.1.2 is not always true the other way around because a *N-TS* may be obtainable from a *GTS* by the exclusion of the null set or the whole set but the same is not the case the other way around. That is, we cannot obtain a *GTS* by including the null set or the whole set to any random *N-TS*. For a function to be weakly Nu-continuous, all the properties of continuity of the function in a *GTS* are intact, except for the exclusion of the null set or the whole set from the *GTSs* in context. However, in other *N-TSs*, where union or intersections of members are not members of a *N-T*, the properties of weakly continuity will fail and hence the converse part will fail in general.

Proposition 5.1.3

If a map is continuous then it is also weakly Nu-continuous.

Proof:

Let f be a map between $\mathcal{T}_1$ and $\mathcal{T}_2$. If f is continuous, then for each $\mathcal{W} \in \mathcal{T}_2$, $f^{-1}(\mathcal{W}) \in \mathcal{T}_1$. Thus, $f^{-1}(\mathcal{X}) \in \mathcal{T}_1$and $f^{-1}(\emptyset) \in \mathcal{T}_1$. Now, the map f maps $(\mathcal{X}, \mathcal{C})$ to $(\mathcal{Y}, \mathcal{C})$ in such a manner that either $\emptyset$ or $\mathcal{X}$ are excluded from the two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. However, the other open sets of the two topologies are intact in $\mathcal{T}_1$ and $\mathcal{T}_2$. Hence, the property of continuity of the map f between $\mathcal{T}_1$ and $\mathcal{T}_2$, is carried over to the map f between $(\mathcal{X}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{C})$ and hence f becomes weakly Nu-continuous.

Remark 5.1.3

The converse of ***Proposition 5.1.3*** is not always true. If f that maps a *GTS* $(\mathcal{X}, \mathcal{T}_1)$ to another *GTS* $(\mathcal{Y}, \mathcal{T}_2)$ is continuous then $f^{-1}(\mathcal{X}) \in \mathcal{T}_1$but if we consider the *N-Ts* $\mathcal{T}_1 \setminus \mathcal{X}$ and $\mathcal{T}_2 \setminus \mathcal{X}$for weakly Nu-continuity then we do not need to worry whether $f^{-1}(\mathcal{X}) \in \mathcal{T}_1$ or not, as $\mathcal{X}$ itself being excluded there will no image of $\mathcal{X}$ in $\mathcal{T}_2$ and as such the map will be weakly Nu-continuous. However, the map will not be continuous because $f^{-1}(\mathcal{X}) \notin \mathcal{T}_1$ as the image of $\mathcal{X}$ will not be there in $\mathcal{T}_2$.

Proposition 5.1.4

For two N-TSs $(\mathcal{X}, \mathcal{T}_1)$, $(\mathcal{Y}, \mathcal{T}_2)$, the map f between $\mathcal{T}_1$ and $\mathcal{T}_2$ will be weakly Nu-continuous iff for every $\mathcal{V}$, a $\mathcal{T}_2$-N-CS, $f^{-1}(\mathcal{V})$ is $\mathcal{T}_1$-N-C.

Proof:

If the map f between $\mathcal{T}_1$ and $\mathcal{T}_2$ is weakly Nu-continuous and $\mathcal{V}$ be any$\mathcal{T}_2$-*N-CS* then $c\mathcal{V}(= \mathcal{Y} \setminus \mathcal{V})$ will be $\mathcal{T}_2$-*N-O* and f being weakly Nu-continuous, $f^{-1}(\mathcal{Y} \setminus \mathcal{V})$ will be $\mathcal{T}_1$-*N-O.*

Now, $f^{-1}(\mathcal{Y} \setminus \mathcal{V}) = \mathcal{X} \setminus f^{-1}(\mathcal{V})$, which is $\mathcal{T}_1$-*N-O* and hence $f^{-1}(\mathcal{V})$ is $\mathcal{T}_1$-*N-C.*

Conversely, for $f^{-1}(\mathcal{V})$ is $\mathcal{T}_1$-*N-C* for every $\mathcal{V}$ that are *N-C* in $\mathcal{T}_2$, then for any $\mathcal{W}$, which is $\mathcal{T}_2$-*N-O*, $\mathcal{Y} \setminus \mathcal{W}$ will be $\mathcal{T}_2$-*N-C* and as such $f^{-1}(\mathcal{Y} \setminus \mathcal{W})$ is $\mathcal{T}_1$-*N-C.*

Now, $f^{-1}(\mathcal{Y} \setminus \mathcal{W}) = \mathcal{X} \setminus f^{-1}(\mathcal{W})$will be $\mathcal{T}_1$-*N-C*, thereby showing that $f^{-1}(\mathcal{W})$is $\mathcal{T}_1$-*N-O*. Hence, as per ***proposition 5.1.1,*** f is weakly Nu-continuous.

Proposition 5.1.5

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, the map f from $\mathcal{T}_1$ to $\mathcal{T}_2$ will be Nu-continuous iff for any $x \in \mathcal{X}$, the pre-image of all $\mathcal{T}_2$-Nu-nhd of $f(x)$ will be $\mathcal{T}_1$-Nu-nhd of x.

Proof:

We assume the map f to be Nu-continuous, and $x \in \mathcal{X}$ and $\mathcal{N}$ be a random $\mathcal{T}_2$-Nu-nhd of $f(x)$. Then the definition of Nu-nhd says that there is a $\mathcal{V} \in \mathcal{T}_2$ so that $f(x) \in \mathcal{V} \subseteq \mathcal{W}$which gives $x \in f^{-1}(\mathcal{V}) \subseteq f^{-1}(\mathcal{W})$. Now, $f$being Nu-continuous so $f^{-1}(\mathcal{W}) \in \mathcal{T}_1$ and since $x \in f^{-1}(\mathcal{V}) \subseteq f^{-1}(\mathcal{W})$ it means that $f^{-1}(\mathcal{W})$ is a $\mathcal{T}_1$-Nu-nhd of x.

Conversely, let $f^{-1}(\mathcal{W})$ be a $\mathcal{T}_1$-Nu-nhd of x for every $\mathcal{T}_2$-Nu-nhd $\mathcal{W}$of $f(x)$, then if $\mathcal{U} \in \mathcal{T}_2$will lead to $x \in f^{-1}(\mathcal{U})$ so that $f(x) \in \mathcal{U}$. Now, since $\mathcal{U} \in \mathcal{T}_2$, it is a $\mathcal{T}_2$-Nu-nhd of $f(x)$and hence by the condition $f^{-1}(\mathcal{U})$ is a $\mathcal{T}_1$-Nu-nhd of $x$and hence $f^{-1}(\mathcal{U}) \in \mathcal{T}_1$ and hence by ***proposition 5.1.1,*** f is Nu-continuous.

Proposition 5.1.6

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, the map f from $\mathcal{T}_1$ to $\mathcal{T}_2$ will be weakly Nu-continuous iff the pre-image of each member of a Nu-sub-base of $\mathcal{Y}$ is N-O in $\mathcal{T}_1$.

Proof:

Let f be weakly Nu-continuous with $\mathcal{B}^s$ being a Nu-sub-base for $\mathcal{Y}$ and let $\mathcal{Q} \in \mathcal{T}_2$. Since each member of $\mathcal{B}^s$ is *N-O* in $\mathcal{T}_2$, so by ***proposition 5.1.1*** it can be concluded that $f^{-1}(\mathcal{Q})$ is *N-O* in $\mathcal{T}_1$ for every $\mathcal{Q} \in \mathcal{B}^s$.

Conversely, let $f^{-1}(\mathcal{Q})$ be *N-O* in $\mathcal{T}_1$ for every $\mathcal{Q} \in \mathcal{B}^s$, and if $\mathcal{P}$ is any *N-OS* in $\mathcal{T}_2$ and $\mathcal{B}$ is a class of all finite intersections of components of $\mathcal{B}^s$so that $\mathcal{B}$ forms a Nu-base for $\mathcal{Y}$ then if $B \in \mathcal{B}$, then there exists finite number of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \ldots, \mathcal{Q}_n$ in $\mathcal{B}^s$ so that $B = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \ldots \cap \mathcal{Q}_n$. Then $f^{-1}(B) = f^{-1}(\mathcal{Q}_1) \cap f^{-1}(\mathcal{Q}_2) \cap \ldots \cap f^{-1}(\mathcal{Q}_n)$. Now, each $f^{-1}(\mathcal{Q}_i) \in \mathcal{T}_1$ so $f^{-1}(B) \in \mathcal{T}_1$.

Also, since $\mathcal{B}$ is a Nu-base for $\mathcal{Y}$, $\mathcal{P} = \cup \{B: B \in \mathcal{B}; B \subseteq \mathcal{P}\}$.

Then $f^{-1}(\mathcal{P}) = f^{-1}[\cup \{B: B \in \mathcal{B}; B \subseteq \mathcal{P}\}] = \cup [f^{-1}(B): B \in \mathcal{B}; B \subseteq \mathcal{P}]$ which is *N-O* in $\mathcal{T}_1$ since each $f^{-1}(B) \in \mathcal{T}_1$. Thus $f^{-1}(\mathcal{P}) \in \mathcal{T}_1$ for each *N-OS* $\mathcal{P}$in $\mathcal{T}_2$. Hence as per ***propositions 5.1.1*** the function f is weakly Nu-continuous.

Remark 5.1.4

In ***proposition 5.1.6*** the map f will not be Nu-continuous because in a *N-TS*, the union of members of the Nu-base may not be *N-O*.

Proposition 5.1.7

For two N-TSs $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, a function f from $\mathcal{T}_1$ to $\mathcal{T}_2$ is weakly Nu-continuous iff the pre-image of every class of a Nu-base for $\mathcal{Y}$ is N-O in $\mathcal{T}_1$.

Proof:

Assume f to be weakly Nu-continuous, and assume B to be any member of a Nu-base $\mathcal{B}$ for $\mathcal{Y}$. Now, B is *N-O* in $\mathcal{T}_2$ since $B \in \mathcal{B} \subseteq \mathcal{T}_2$ and hence by ***proposition 5.1.1,*** $f^{-1}(B) \in \mathcal{T}_1$.

Conversely, let $f^{-1}(B)$ is *N-O* member of $\mathcal{T}_1$ for any $B \in \mathcal{B}$ and assume $\mathcal{O}$ to be any *N-OS* in $\mathcal{T}_2$, then $\mathcal{O}$ can be described as: $\mathcal{O} = \cup \{B: B \in \mathcal{B}; B \subseteq \mathcal{O}\}$.

Hence $f^{-1}(\mathcal{O}) = f^{-1}[\cup \{B: B \in \mathcal{B}; B \subseteq \mathcal{O}\}] = \cup [f^{-1}(B): B \in \mathcal{B}; B \subseteq \mathcal{O}]$ which is *N-O* since each $f^{-1}(B)$ is *N-O*. Hence as per ***proposition 5.1.1,*** f is weakly Nu-continuous.

Remark 5.1.5

In ***proposition 5.1.7,*** f will not be Nu-continuous because in a *N-TS,* the union of members of the Nu-base may not be *N-O.*

Proposition 5.1.8

For two N-TSs $(\mathcal{X}, \mathcal{T}_1)$ *and* $(\mathcal{Y}, \mathcal{T}_2)$, *a function* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is weakly Nu-continuous iff* $(f^{-1}(\mathcal{B}))^{Nu-cl} \subseteq f^{-1}(\mathcal{B}^{Nu-cl})$ *for any subset* $\mathcal{B}$ *of* $\mathcal{Y}$.

Proof:

Assume f to be weakly Nu-continuous, then $\mathcal{B}^{Nu-cl}$ is *N-C* with respect to $\mathcal{T}_2$ and so by ***proposition 5.1.4***, $f^{-1}(\mathcal{B}^{Nu-cl})$ is *N-C* with respect to $\mathcal{T}_1$ and hence $[f^{-1}(\mathcal{B}^{Nu-cl})]^{Nu-cl} = f^{-1}(\mathcal{B}^{Nu-cl})$.

Now, $\mathcal{B} \subseteq \mathcal{B}^{Nu-cl}$ and so, $f^{-1}[\mathcal{B}] \subseteq f^{-1}[\mathcal{B}^{Nu-cl}]$

$\Rightarrow [f^{-1}(\mathcal{B})]^{Nu-cl} \subseteq [f^{-1}(\mathcal{B}^{Nu-cl})]^{Nu-cl}$, [by ***proposition 2.3.3 (iii)***]

But $[f^{-1}(\mathcal{B}^{Nu-cl})]^{Nu-cl} = f^{-1}(\mathcal{B}^{Nu-cl})$, so $(f^{-1}(\mathcal{B}))^{Nu-cl} \subseteq f^{-1}(\mathcal{B}^{Nu-cl})$.

Conversely, let the condition be true. Now, if $\mathcal{C}$ be any *N-CS* in $\mathcal{Y}$ then $\mathcal{C}^{Nu-cl} = \mathcal{C}$.

Now, by condition $(f^{-1}(\mathcal{C}))^{Nu-cl} \subseteq f^{-1}(\mathcal{C}^{Nu-cl}) = f^{-1}(\mathcal{C})$

That is, $(f^{-1}(\mathcal{C}))^{Nu-cl} \subseteq f^{-1}(\mathcal{C})$.

But, $f^{-1}(\mathcal{C}) \subseteq (f^{-1}(\mathcal{C}))^{Nu-cl}$, [by ***proposition 2.3.3 (i)***]

Hence $(f^{-1}(\mathcal{C}))^{Nu-cl} = f^{-1}(\mathcal{C})$, thus showing that $f^{-1}(\mathcal{C})$ is *N-C* in $\mathcal{T}_1$ and hence by ***proposition 5.1.4***, the function f is weakly Nu-continuous.

Remark 5.1.6

In ***proposition 5.1.8***, the function f will not be Nu-continuous because in a *N-TS,* the Nu-closure of a set is not necessarily a *N-CS* [by ***remark 2.3.1***].

Proposition 5.1.9

For two N-TS $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is weakly Nu-continuous iff* $f(\mathcal{C}^{Nu-cl}) \subseteq [f(\mathcal{C})]^{Nu-cl}$ *for each subset* $\mathcal{C}$ *of* $\mathcal{X}$.

Proof:

Let f be weakly Nu-continuous and $\mathcal{C} \subseteq \mathcal{X}$ and let $f(\mathcal{C}) = \mathcal{B} \subseteq \mathcal{Y}$. Then by ***proposition 5.1.8*** we have $(f^{-1}(\mathcal{B}))^{Nu-cl} \subseteq f^{-1}(\mathcal{B}^{Nu-cl})$

$$\Rightarrow [f^{-1}(f(\mathcal{C}))]^{Nu-cl} \subseteq f^{-1}[f(\mathcal{C}))^{Nu-cl}]$$

$\Rightarrow f^{-1}(f(\mathcal{C}^{Nu-cl})) \subseteq f^{-1}[(f(\mathcal{C}))^{Nu-cl}]$, *since* $f(\mathcal{C}) \subseteq f(\mathcal{C}^{Nu-cl})$

Thus, $f(\mathcal{C}^{Nu-cl}) \subseteq [f(\mathcal{C})]^{Nu-cl}$

Conversely, let the condition be true and assume that $\mathcal{B}$ is some arbitrary *N-CS* set in $\mathcal{Y}$, then $f^{-1}(\mathcal{B}) \subseteq \mathcal{X}$.

Now, by the condition, $f((f^{-1}(\mathcal{B}))^{Nu-cl}) \subseteq [f(f^{-1}(\mathcal{B}))]^{Nu-cl}$

$$\Rightarrow f((f^{-1}(\mathcal{B}))^{Nu-cl}) \subseteq f(f^{-1}(\mathcal{B}^{Nu-cl}))$$

$$\Rightarrow (f^{-1}(\mathcal{B}))^{Nu-cl} \subseteq f^{-1}(\mathcal{B}), \text{ since } \mathcal{B} \text{ is } N\text{-}CS$$

But, $f^{-1}(\mathcal{B}) \subseteq (f^{-1}(\mathcal{B}))^{Nu-cl}$, [by ***proposition 2.3.3 (i)***]

Hence, we get $(f^{-1}(\mathcal{B}))^{Nu-cl} = f^{-1}(\mathcal{B})$ thereby showing $f^{-1}(\mathcal{B})$ is *N-C* in $\mathcal{T}_1$ and hence by ***proposition 5.1.4***, f is weakly Nu-continuous.

Remark 5.1.7

In ***proposition 5.1.9,*** f will not be Nu-continuous because the Nu-closure of a set being equal to the set does not always mean that the set is *N-C* [by ***remark 2.3.1***].

Proposition 5.1.10

For two N-TS $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is weakly Nu-continuous iff* $f^{-1}(\mathcal{A}^{Nu-int}) \subseteq [f^{-1}(\mathcal{A})]^{Nu-int}$ *for any subset* $\mathcal{A}$ *of* $\mathcal{Y}$.

Proof:

Let f be weakly Nu-continuous. Now since $\mathcal{A}^{Nu-int}$ is *N-O* in $\mathcal{Y}$ so by ***proposition 5.1.1***, $f^{-1}(\mathcal{A}^{Nu-int})$ is *N-O* in $\mathcal{X}$ and so $[f^{-1}(\mathcal{A}^{Nu-int})]^{Nu-int} = f^{-1}(\mathcal{A}^{Nu-int})$.

Now, $\mathcal{A}^{Nu-int} \subseteq \mathcal{A} \Rightarrow f^{-1}(\mathcal{A}^{Nu-int}) \subseteq f^{-1}(\mathcal{A})$

$\Rightarrow [f^{-1}(\mathcal{A}^{Nu-int})]^{Nu-int} \subseteq [f^{-1}(\mathcal{A})]^{Nu-int}$

Or, $f^{-1}(\mathcal{A}^{Nu-int}) \subseteq [f^{-1}(\mathcal{A})]^{Nu-int}$, since $f^{-1}(\mathcal{A}^{Nu-int})$ is *N-O* in $\mathcal{X}$.

Conversely, if the condition is true then let $\mathcal{B}$ be any *N-OS* in $\mathcal{Y}$ so that $\mathcal{B}^{Nu-int} = \mathcal{B}$, then by the condition $f^{-1}\left(\mathcal{B}^{Nu-int}\right) \subseteq [f^{-1}(\mathcal{B})]^{Nu-int}$, or $f^{-1}(\mathcal{B}) \subseteq [f^{-1}(\mathcal{B})]^{Nu-int}$. But, we have, in general $[f^{-1}(\mathcal{B})]^{Nu-int} \subseteq f^{-1}(\mathcal{B})$ and so $[f^{-1}(\mathcal{B})]^{Nu-int} = f^{-1}(\mathcal{B})$ which means that $f^{-1}(\mathcal{B})$ is *N-O* in $\mathcal{T}_1$ and hence by ***proposition 5.1.1,*** f is weakly Nu-continuous.

Remark 5.1.8

In ***proposition 5.1.10***, the mapping f will not be Nu-continuous because in a *N-TS*, the Nu-interior of a set is not necessarily a *N-OS*. [by ***Remark 2.1.3***]

Proposition 5.1.11

For three N-TS $(\mathcal{X}, \mathcal{T}_1)$, $(\mathcal{Y}, \mathcal{T}_2)$, *and* $(\mathcal{Z}, \mathcal{T}_3)$ *if the maps* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *and* g *from* $\mathcal{T}_2$ *to* $\mathcal{T}_3$ *are Nu-continuous, then the map from* $(\mathcal{X}, \mathcal{T}_1)$ *to* $(\mathcal{Z}, \mathcal{T}_3)$ *given by:* $g \circ f : (\mathcal{X}, \mathcal{T}_1) \to (\mathcal{Z}, \mathcal{T}_3)$ *is also Nu-continuous.*

Proof:

Assume that $\mathcal{C}$ is a *N-OS* in $\mathcal{T}_3$, then by ***proposition 5.1.1,*** $g^{-1}(\mathcal{C})$ is *N-O* in $\mathcal{T}_2$ and by the same proposition $f^{-1}[g^{-1}(\mathcal{C})]$ is *N-O* in $\mathcal{T}_1$.

But $f^{-1}[g^{-1}(\mathcal{C})] = [f^{-1} \circ g^{-1}](\mathcal{C}) = (g \circ f)^{-1}(\mathcal{C})$. Thus, the pre-image with respect to $(g \circ f)$ of all sets that are *N-O* in $\mathcal{T}_3$ are also *N-O* in $\mathcal{T}_1$ and hence by ***proposition 5.1.1,*** $g \circ f$ is Nu-continuous.

Proposition 5.1.12

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ *and* $(\mathcal{Y}, \mathcal{T}_2)$, *if* $\mathcal{A}$ *is some non-null subset of* $\mathcal{X}$ *and if* $f : (\mathcal{X}, \mathcal{T}_1) \to (\mathcal{Y}, \mathcal{T}_2)$ *is weakly Nu-continuous, then the function* $f_{\mathcal{A}} : \mathcal{A} \to \mathcal{Y}$ *is weakly Nu-continuous.*

Proof:

Assume $\mathcal{B}$ to be *N-O* in $\mathcal{Y}$, then by definition, we have: ${f_{\mathcal{A}}}^{-1}(\mathcal{B}) = \mathcal{A} \cap f^{-1}(\mathcal{B})$.

Now, since f is weakly Nu-continuous, by ***proposition 5.1.1,*** $f^{-1}(\mathcal{B})$ is *N-O* in $\mathcal{T}_1$ and hence $\mathcal{A} \cap f^{-1}(\mathcal{B})$ is *N-O* in $\mathcal{A}$ and by ***proposition 5.1.1,*** $f_{\mathcal{A}}$ is weakly Nu-continuous.

Proposition 5.1.13

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$, $(\mathcal{Y}, \mathcal{T}_2)$, *and* $\{x\}$ *a singleton subset of* $\mathcal{X}$, *the function* $f : (\mathcal{X}, \mathcal{T}_1) \to (\mathcal{Y}, \mathcal{T}_2)$ *is Nu-continuous at* $x \in \mathcal{X}$.

Proof:

Let $\mathcal{B}$ be any *N-O* subset of $\mathcal{Y}$ and let $f(x) \in \mathcal{B}$.

Now, $f(x) \in \mathcal{B} \Rightarrow x \in f^{-1}(\mathcal{B})$

$\Rightarrow \{x\} \in f^{-1}(\mathcal{B})$

$\Rightarrow f$ is Nu-continuous at the point $x \in \mathcal{X}$.

Proposition 5.1.14

For a N-TS $(\mathcal{X}, \mathcal{C})$, the identity map $f: \mathcal{X} \to \mathcal{X}$, defined as $f(x) = x$ for every $x \in \mathcal{X}$ is Nu-continuous.

Proof:

Let $\mathcal{B} \in \mathcal{T}$, *i.e.* $\mathcal{B} \subseteq \mathcal{X}$. Now, $f(x) = x \in \mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{X}$

$$\Rightarrow f^{-1}(\mathcal{B}) = \{x \in \mathcal{X}: \zeta(x) \in \mathcal{B}\}$$
$$\Rightarrow f^{-1}(\mathcal{B}) = \{x \in \mathcal{X}: x \in \mathcal{B}\}$$
$$\Rightarrow f^{-1}(\mathcal{B}) = \{x\}$$
$$\Rightarrow f^{-1}(\mathcal{B}) \text{ is } N\text{-}O \text{ in } \mathcal{X}.$$
$$\Rightarrow f \text{ is Nu-continuous.}$$

Definition 5.1.3

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, a map $f: \mathcal{X} \to \mathcal{Y}$ is called a N-O map if the images of all $\mathcal{T}_1$ N-OS are N-OS in $\mathcal{T}_2$. The function f will be called Nu-bi-continuous if it is Nu-continuous and a N-O map.

A map $f: \mathcal{X} \to \mathcal{Y}$ is called a N-C map if the images of all $\mathcal{T}_1$ N-CSs are N-CSs in $\mathcal{T}_2$.

Definition 5.1.4

If $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$ be two N-TSs, then a mapping f of $\mathcal{X}$ into $\mathcal{Y}$ is said to be a Nu-homeomorphism if:

- *(i)* f *is one-one and onto*
- *(ii)* $f: \mathcal{X} \to \mathcal{Y}$ *is weakly Nu-continuous.*
- *(iii)* $f^{-1}: \mathcal{Y} \to \mathcal{X}$ *is weakly Nu-continuous.*

If such a function f exists then $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$ are said to be Nu-homeomorphic to each other.

Proposition 5.1.15

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, if f is one-one and onto mapping of $\mathcal{X}$ to $\mathcal{Y}$, then f is a Nu-homeomorphism iff f is weakly Nu-continuous and N-O map.

Proof:

Assume f is a Nu-homeomorphism and let $f^{-1} = g$ and $g^{-1} = f$. Now, we have f is one-one onto, and also g is one-one onto. Let $\mathcal{O} \in \mathcal{T}_1$, then $g^{-1}(\mathcal{O}) \in \mathcal{T}_2$. But since $g^{-1} = f$ so $g^{-1}(\mathcal{O}) = f(\mathcal{O}) \in \mathcal{T}_2$. Since $\mathcal{O} \in \mathcal{T}_1$ and $f(\mathcal{O}) \in \mathcal{T}_2$, it follows that f is a *N-O* mapping and by virtue of Nu-homeomorphism, f is weakly Nu-continuous.

Conversely, let f is weakly Nu-continuous and a *N-O* map. Also, by condition f is one-one onto. Suffices to prove that $f^{-1} = g$ is weakly Nu-continuous. Let $\mathcal{O} \in \mathcal{T}_1$, then $f(\mathcal{O}) \in \mathcal{T}_2$ since f is a *N-O* map. That is, $g^{-1}(\mathcal{O}) \in \mathcal{T}_2$ thereby showing that $g = f^{-1}$ is weakly Nu-continuous. Hence f is a Nu-homeomorphism.

Proposition 5.1.16

For two N-TS $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, if f is one-one and onto mapping of $\mathcal{X}$ to $\mathcal{Y}$, then f is a Nu-homeomorphism if and only if f is weakly Nu-continuous and N-C map.

Proof:

Let f be a Nu-homeomorphism and let $\mathcal{C}$ be any $\mathcal{T}_1$-*N-CS*. Then $\mathcal{X} \setminus \mathcal{C}$is *N-OS* in $\mathcal{T}_1$. Since $g = f^{-1}$ is weakly Nu-continuous, it follows that $g^{-1}(\mathcal{X} \setminus C)$ is *N-OS* in $\mathcal{T}_2$. But, $g^{-1}(\mathcal{X} \setminus C) = \mathcal{Y} \setminus g^{-1}(\mathcal{C})$. Hence $\mathcal{Y} \setminus g^{-1}(\mathcal{C})$ is *N-OS* in $\mathcal{T}_2$ and as such $g^{-1}(\mathcal{C})$ is *N-CS* in $\mathcal{T}_2$, that is $g^{-1}(\mathcal{C}) = f(\mathcal{C})$ is *N-CS* in $\mathcal{T}_2$. Hence f is weakly Nu-continuous and a *N-C* map.

Conversely, let the conditions hold and let $\mathcal{O}$ be any *N-OS* in $\mathcal{T}_1$, then $\mathcal{X} \setminus \mathcal{O}$ is *N-CS* and since f is a *NC* map, $f(\mathcal{X} \setminus \mathcal{O}) = g^{-1}(\mathcal{X} \setminus \mathcal{O}) = \mathcal{Y} \setminus g^{-1}(\mathcal{O})$ is a *N-CS* in $\mathcal{T}_2$ which implies that $g^{-1}(\mathcal{O})$ is *N-OS* in $\mathcal{T}_2$. Thus, pre-image of every *N-OS* in $\mathcal{T}_1$ under the function g is *N-OS* in $\mathcal{T}_2$. Thus, $g = f^{-1}$ is weakly Nu-continuous and hence f is a Nu-homeomorphism.

Proposition 5.1.17

For two N-TSs $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{Y}, \mathcal{T}_2)$, if a mapping f from $\mathcal{T}_1$ to $\mathcal{T}_2$ is one-one onto and weakly Nu-continuous then f is a Nu-homeomorphism if f is N-O or N-C map.

Proof:

We assume that f is one-one onto and weakly Nu-continuous and also that f is either a *N-O* or *N-C* map. We will show that f^{-1} is weakly Nu-continuous. It will suffice to show that $f^{-1}(\mathcal{B}^{Nu-cl}) \subseteq [f^{-1}(\mathcal{B})]^{Nu-cl}$ as per ***proposition 5.1.9*** for any $\mathcal{B} \subseteq \mathcal{Y}$.

Now, $\mathcal{B} \subseteq \mathcal{Y} \Rightarrow [f^{-1}(\mathcal{B})]^{Nu-cl} \subseteq \mathcal{X}$ and is a *N-CS* in $\mathcal{X}$.

And since f is a *N-C* map, we have:

$f([f^{-1}(\mathcal{B})]^{Nu-cl}) = \{f([f^{-1}(\mathcal{B})]^{Nu-cl})\}^{Nu-cl}$, since $f(\mathcal{A}) = [f(\mathcal{A})]^{Nu-cl}$........... *(1)*

Now, $f^{-1}(\mathcal{B}) \subseteq [f^{-1}(\mathcal{B})]^{Nu-cl}$

This implies: $f(f^{-1}(\mathcal{B})) \subseteq f([f^{-1}(\mathcal{B})]^{Nu-cl})$

$\Rightarrow [f(f^{-1}(\mathcal{B}))]^{Nu-cl} \subseteq [f([f^{-1}(\mathcal{B})]^{Nu-cl})]^{Nu-cl}$

$\Rightarrow [f(f^{-1}(\mathcal{B}))]^{Nu-cl} \subseteq f([f^{-1}(\mathcal{B})]^{Nu-cl})$ using *(1)*

$\Rightarrow f\left(f^{-1}(\mathcal{B}^{Nu-cl})\right) \subseteq f([f^{-1}(\mathcal{B})]^{Nu-cl})$

$\Rightarrow f^{-1}(\mathcal{B}^{Nu-cl}) \subseteq [f^{-1}(\mathcal{B})]^{Nu-cl}$

$\Rightarrow f^{-1}$ is weakly Nu-continuous by ***proposition 5.1.9***. Hence the function f is a Nu-homeomorphism.

Proposition 5.1.18

For two N-TS $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ *and* $(\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$, *a function* $f: (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$ *is N-O iff* $f\left(\mathcal{A}^{Nu-int}\right) \subseteq [f(\mathcal{A})]^{Nu-int}$ *for every* $\mathcal{A} \subseteq \mathcal{X}$.

Proof:

Let f be *N-O* map and $\mathcal{A} \subseteq \mathcal{X}$ then $f(\mathcal{A}^{Nu-int})$ is *N-O* in $\mathcal{C}_{\mathcal{Y}}$ since $\mathcal{A}^{Nu-int}$ is *N-O* in $\mathcal{C}_{\mathcal{X}}$. Now, $\mathcal{A}^{Nu-int} \subseteq \mathcal{A}$, so $f(\mathcal{A}^{Nu-int}) \subseteq f(\mathcal{A})$. Again, since $f(\mathcal{A}^{Nu-int})$is *N-O* in $\mathcal{C}_{\mathcal{Y}}$, so $[f(\mathcal{A}^{Nu-int})]^{Nu-int} = f(\mathcal{A}^{Nu-int}$ …………… *(1)*

Also, $f\left(\mathcal{A}^{Nu-int}\right) \subseteq f(\mathcal{A}) \Rightarrow [f\left(\mathcal{A}^{Nu-int}\right)]^{Nu-int} \subseteq [f(\mathcal{A})]^{Nu-int}$

$\Rightarrow f(\mathcal{A}^{Nu-int}) \subseteq [f(\mathcal{A})]^{Nu-int}$ by *(1)*

Conversely, let the condition be true. That is, $f(\mathcal{A}^{Nu-int}) \subseteq [f(\mathcal{A})]^{Nu-int}$ for every $\mathcal{A} \subseteq \mathcal{X}$ and let $\mathcal{O}$ be any set in $\mathcal{C}_{\mathcal{X}}$, so that $\mathcal{O}^{Nu-int} = \mathcal{O}$.

Then $f(\mathcal{O}) = f\left(\mathcal{O}^{Nu-int}\right) \subseteq [f(\mathcal{O})]^{Nu-int}$, by the assumed condition.

But, in general $[f(\mathcal{O})]^{Nu-int} \subseteq f(\mathcal{O})$.

Thus, we have: $[f(\mathcal{O})]^{Nu-int} = f(\mathcal{O})$, thereby showing that $f(\mathcal{O})$is *N-O* in $\mathcal{C}_{\mathcal{Y}}$ which leads to the conclusion that f is a *N-O* map.

Proposition 5.1.19

For two N-TS $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ *and* $(\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$, *a mapping* $f: (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$ *is N-C map iff* $[f(\mathcal{C})]^{Nu-cl} \subseteq f(\mathcal{C}^{Nu-cl})$ *for every* $\mathcal{C} \subseteq \mathcal{X}$.

Proof:

Let f be *N-C* map and $\mathcal{C} \subseteq \mathcal{X}$. Since $\mathcal{C}^{Nu-cl}$ is *N-C* in $\mathcal{T}_{\mathcal{X}}$ and f is a *N-C* map we have $f(\mathcal{C}^{Nu-cl})$ is *N-C* in $\mathcal{T}_{\mathcal{Y}}$ and consequently, we have:

$[f(\mathcal{C}^{Nu-cl})]^{Nt-cl} = f(\mathcal{C}^{Nu-cl})$ .... *(1)*

Again, $\mathcal{C} \subseteq \mathcal{C}^{Nu-cl} \Rightarrow f(\mathcal{C}) \subseteq f(\mathcal{C}^{Nu-cl})$

$\Rightarrow [f(\mathcal{C})]^{Nu-cl} \subseteq [f(\mathcal{C}^{Nu-cl})]^{Nu-cl} = f(\mathcal{C}^{Nu-cl})$ *by (1)*

Thus, $[f(\mathcal{C})]^{Nu-cl} \subseteq f(\mathcal{C}^{Nu-cl})$.

Conversely, let $[f(\mathcal{C})]^{Nu-cl} \subseteq f(\mathcal{C}^{Nu-cl})$ for all $\mathcal{C} \subseteq \mathcal{X}$ and if $\mathcal{D}$ be any $\mathcal{T}_{\mathcal{X}}$ *N-CS* so that $\mathcal{D}^{Nu-Cl} = \mathcal{D}$. Then $f(\mathcal{D}^{Nu-Cl}) = f(\mathcal{D})$ .... *(2)*

Now, by condition $[f(\mathcal{D})]^{Nu-cl} \subseteq f(\mathcal{D}^{Nu-cl}) = f(\mathcal{D})$ by *(2)*

Thus, $[f(\mathcal{D})]^{Nu-cl} \subseteq f(\mathcal{D})$

But in general, $f(\mathcal{D}) \subseteq [f(\mathcal{D})]^{Nu-cl}$, since $\mathcal{A} \subseteq \mathcal{A}^{Nu-cl}$, [by ***proposition 2.3.3 (i)***]

Thus, we have $[f(\mathcal{D})]^{Nu-cl} = f(\mathcal{D})$, thereby showing that $f(\mathcal{D})$ is *N-CS* in $\mathcal{T}_{\mathcal{Y}}$.

Hence f is a *N-C* map.

Proposition 5.1.20

For two N-TS $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$, if the map $f: (\mathcal{X}, \mathcal{T}_{\mathcal{X}}) \rightarrow (\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ be one-one onto, then f is a Nu-homeomorphism if and only if $[f(\mathcal{C})]^{Nu-cl} = f(\mathcal{C}^{Nu-cl})$ for all $\mathcal{C} \subseteq \mathcal{X}$.

Proof:

Let f be a Nu-homeomorphism. Then f is one-one onto, f is weakly Nu-continuous and f is *N-C*, by ***proposition 5.1.17***.

Then by ***proposition 5.1.19,*** we have: $f(\mathcal{C}^{Nu-cl}) \subseteq [f(\mathcal{C})]^{Nu-cl}$ .... *(1)*

Also $\mathcal{C} \subseteq \mathcal{C}^{Nu-cl} \Rightarrow f(\mathcal{C}) \subseteq f(\mathcal{C}^{Nu-cl}) \Rightarrow [f(\mathcal{C})]^{Nu-cl} \subseteq [f(\mathcal{C}^{Nu-cl})]^{Nu-cl}$............ *(2)*

Now, f is a *N-C* map and $\mathcal{C}^{Nu-cl}$ is *N-CS* in $\mathcal{T}_{\mathcal{X}}$ and hence $f(\mathcal{C}^{Nu-cl})$ is *N-CS* in $\mathcal{T}_{\mathcal{Y}}$.

Hence $[f(\mathcal{C}^{Nu-cl})]^{Nu-cl} = f(\mathcal{C}^{Nu-cl})$ .... *(3)*

From *(2)* and *(3)*, we get $[f(\mathcal{C})]^{Nu-cl} \subseteq f(\mathcal{C}^{Nu-cl})$ .... *(4)*

From *(1)* and *(4)*, we have: $f(\mathcal{C}^{Nu-cl}) = [f(\mathcal{C})]^{Nu-cl}$.

Conversely, let $f(\mathcal{C}^{Nu-cl}) = [f(\mathcal{C})]^{Nu-cl}$ for every $\mathcal{C} \subseteq \mathcal{X}$.

Then obviously $f(\mathcal{C}^{Nu-cl}) \subseteq [f(\mathcal{C})]^{Nu-cl}$, and so by ***proposition 5.1.19,*** the function f is weakly Nu-continuous.

Again, if $\mathcal{D}$ is any *N-CS* in $\mathcal{T}_{\mathcal{X}}$, so that $\mathcal{D}^{Nu-cl} = \mathcal{D}$, then $f(\mathcal{D}^{Nu-cl}) = f(\mathcal{D})$

$\Rightarrow f(\mathcal{D}) = f(\mathcal{D}^{Nu-cl}) = [f(\mathcal{D})]^{Nu-cl}$ by the given condition.

Hence $f(\mathcal{D})$ is *N-C* in $\mathcal{C}_{\mathcal{Y}}$ for every *N-CS* $\mathcal{D}$in $\mathcal{C}_{\mathcal{X}}$ and so the function f is *N-C*.

Now, since f is *N-C* as well as Nu-continuous and it is also given to be one-one and onto and hence f is a Nu-homeomorphism.

Proposition 5.1.21

For two N-TS $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ and $\left(\mathcal{Y}, \mathcal{C}_{\mathcal{Y}}\right)$, if the mapping $f: (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$ be N-O and onto, and if $\mathcal{B}$ is a Nu-base for $\mathcal{C}_{\mathcal{X}}$ then the class $\{f(B): B \in \mathcal{B}\}$ is a Nu-base for $\mathcal{C}_{\mathcal{Y}}$.

Proof:

Assume $\mathcal{Q}$ to be any *N-OS* in $\mathcal{C}_{\mathcal{Y}}$ and say $y \in \mathcal{Q}$ be an arbitrary member. Now since f is onto, there will be some *x* so that $f(x) = y$.

Moreover, $\mathcal{B}$ being a Nu-base for $\mathcal{C}_{\mathcal{X}}$, there will be some member of $\mathcal{B}$ to which *x* belongs. If B_x happens to be the smallest member of $\mathcal{B}$ so that $x \in B_x$, then f being *N-O*, $\zeta(B_x)$ will be *N-O* in $\mathcal{C}_{\mathcal{Y}}$. Also, $f(x) \in f(B_x)$ and as such $f(B_x)$ will be the smallest *N-OS* containing B_x in $\mathcal{C}_{\mathcal{Y}}$ since B_x is the smallest *N-OS* containing *x* in $\mathcal{C}_{\mathcal{X}}$.

Thus, we must have: $y = f(x) \in f(B_x) \subseteq \mathcal{Q}$ and since the member B of $\mathcal{B}$ is arbitrary so the class $\{f(B): B \in \mathcal{B}\}$ becomes a Nu-base for $\mathcal{C}_{\mathcal{Y}}$.

Proposition 5.1.22

For two N-TS $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ and $\left(\mathcal{Y}, \mathcal{C}_{\mathcal{Y}}\right)$ let $\mathcal{B}$ be a Nu-base for $\mathcal{C}_{\mathcal{X}}$. If the mapping $f: (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$ be such that $f(\mathcal{B}) \in \mathcal{C}_{\mathcal{Y}}$ for every $B \in \mathcal{B}$, then f is a N-O map.

Proof:

Let $\mathcal{O}$ be any member of $\mathcal{C}_{\mathcal{X}}$. It suffices to show that $f(\mathcal{O})$ is a member of $\mathcal{C}_{\mathcal{Y}}$. Since $\mathcal{B}$ is a Nu-base for $\mathcal{C}_{\mathcal{X}}$we have: $\mathcal{O} = \cup \{B_\alpha: B_\alpha \in \mathcal{B}\}$.

So $f(\mathcal{O}) = f(\cup \{B_\alpha: B_\alpha \in \mathcal{B}\}) = \cup \{f(B_\alpha): B_\alpha \in \mathcal{B}\}$. Now, by the given condition each $f(\mathrm{B}_\alpha) \in \mathcal{C}_{\mathcal{Y}}$ and hence $f(\mathcal{O}) \in \mathcal{C}_{\mathcal{Y}}$ and hence the function f is *N-O*.

Proposition 5.1.23

For two N-TSs $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ and $\left(\mathcal{Y}, \mathcal{C}_{\mathcal{Y}}\right)$, let the mapping $f: (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}})$ be a Nu-homeomorphism then for $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{Y}$ such that $f(\mathcal{A}) = \mathcal{B}$, the map $f_{\mathcal{A}}: \left(\mathcal{X}, \mathcal{C}_{\mathcal{X}/\mathcal{A}}\right) \to (\mathcal{Y}, \mathcal{C}_{\mathcal{Y}/\mathcal{B}})$ is a Nu-homeomorphism, where $\mathcal{C}_{\mathcal{X}/\mathcal{A}}$ and $\mathcal{C}_{\mathcal{Y}/\mathcal{B}}$ denote the relative TSs.

Proof:

Since f is one-one, so $f_{\mathcal{A}}$ is also one-one. Also, since $f(\mathcal{A}) = \mathcal{B}$ we have $f_{\mathcal{A}}(\mathcal{A}) = \mathcal{B}$ thereby showing that $f_{\mathcal{A}}$ is onto also. Next, let $\mathcal{O} \in \mathcal{C}_{\mathcal{X}/\mathcal{A}}$, then $\mathcal{O} = \mathcal{A} \cap \mathcal{P}$, where $\mathcal{P} \in \mathcal{C}_{\mathcal{X}}$. Now since f is one-one, $f(\mathcal{A} \cap \mathcal{P}) = f(\mathcal{A}) \cap f(\mathcal{P})$.

So, $f_{\mathcal{A}}(\mathcal{O}) = f(\mathcal{O}) = f(\mathcal{A}) \cap f(\mathcal{P}) = \mathcal{B} \cap f(\mathcal{P})$

Now, f is *N-O* and $\mathcal{P} \in \mathcal{C}_{\mathcal{X}} \Rightarrow f(\mathcal{P}) \in \mathcal{C}_{\mathcal{Y}}$. Hence $f_{\mathcal{A}}(\mathcal{O}) \in \mathcal{C}_{\mathcal{Y}}$ and so $f_{\mathcal{A}}$ is *N-O* and $f_{\mathcal{A}}$ is Nu-continuous by the Nu-continuity of f, by ***proposition 5.1.1***

Thus, $f_{\mathcal{A}}$ is a Nu-homeomorphism.

5.2 Continuity in Anti-Topological Spaces

Remark 5.2.1

Definition of continuity of functions in an *A-TS* has been provided in ***chapter 1*** in the ***definitions 1.6.24*** and ***1.6.25***.

Proposition 5.2.1

For three A-TS $(\mathcal{X}, \mathcal{T}_1)$, $(\mathcal{Y}, \mathcal{T}_2)$, *and* $(\mathcal{Z}, \mathcal{T}_3)$ *if the functions* f *from* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *and* g *from* $\mathcal{T}_2$ *to* $\mathcal{T}_3$ *are anti-continuous, then the function from* $(\mathcal{X}, \mathcal{T}_1)$ *to* $(\mathcal{Z}, \mathcal{T}_3)$ *which is given by* $g \circ f: (\mathcal{X}, \mathcal{T}_1) \to (\mathcal{Z}, \mathcal{T}_3)$ *is also anti-continuous.*

Proof*:*

Let $\mathcal{C}$ be an *A-OS* in $\mathcal{Z}$, then by ***definition 1.6.24*** $g^{-1}(\mathcal{C})$ is *A-OS* in $\mathcal{Y}$ and by the same definition $f^{-1}[g^{-1}(\mathcal{C})]$ is *A-OS* in $\mathcal{X}$.

But $f^{-1}[g^{-1}(\mathcal{C})] = [f^{-1} \circ g^{-1}](\mathcal{C}) = (g \circ f)^{-1}(\mathcal{C})$. Thus, the pre-image under $g \circ f$ of all *A-OS* in $\mathcal{Z}$ are *A-OS* in $\mathcal{X}$ and hence by ***definition 1.6.24,*** the function $g \circ f$ is anti-continuous.

Proposition 5.2.2

For two A-TS $(\mathcal{X}, \mathcal{T}_1)$, $(\mathcal{Y}, \mathcal{T}_2)$, *and* $\{x\}$ *a singleton subset of* $\mathcal{X}$, *the function* $f: (\mathcal{X}, \mathcal{T}_1) \to (\mathcal{Y}, \mathcal{T}_2)$ *is anti-continuous at* $x \in \mathcal{X}$.

Proof:

Let $\mathcal{B}$ be an *A-O* subset of $\mathcal{Y}$ and let $f(x) \in \mathcal{B}$.

Now, $f(x) \in \mathcal{B} \Rightarrow x \in f^{-1}(\mathcal{B})$

$\Rightarrow \{x\} \in f^{-1}(\mathcal{B})$

$\Rightarrow f$ is anti-continuous at the point $x \in \mathcal{X}$

Proposition 5.2.3

For an A-TS $(\mathcal{X},\mathcal{T})$, the identity map $f:\mathcal{X}\to\mathcal{X}$, defined as $f(x)=x$ for every $x\in\mathcal{X}$ is anti-continuous.

Proof:

Let $\mathcal{B}\in\mathcal{T}$, *i.e.* $\mathcal{B}\subseteq\mathcal{X}$.

Now, $f(x)=x\in\mathcal{X}$ and $\mathcal{B}\subseteq\mathcal{X}$

$\Rightarrow f^{-1}(\mathcal{B})=\{x\in\mathcal{X}: f(x)\in\mathcal{B}\}$

$\Rightarrow f^{-1}(\mathcal{B})=\{x\in\mathcal{X}: x\in\mathcal{B}\}$

$\Rightarrow f^{-1}(\mathcal{B})=\{x\}$

$\Rightarrow f^{-1}(\mathcal{B})$ is *A-O* in $\mathcal{X}$.

$\Rightarrow$ f is anti-continuous.

Proposition 5.2.4

If a function f between two A-TSs $(\mathcal{X},\mathcal{T}_1)$ and $(\mathcal{Y},\mathcal{T}_2)$ is anti-continuous then for each $x\in\mathcal{X}$ and for any A-OS $\mathcal{B}$ containing $f(x)$ there will be an A-OS $\mathcal{A}$ which contains x so that f$(\mathcal{A})=\mathcal{B}$.

Proof:

Assume $f(x)\in\mathcal{B}$, then $x\in f^{-1}(\mathcal{B})$. Now, if f is anti-continuous then $f^{-1}(\mathcal{B})$ is *A-O* in $\mathcal{T}_1$. Now, $\mathcal{A}$ is *A-OS* in $\mathcal{X}$ that contains x and $f^{-1}(\mathcal{B})$ is also *A-OS* in $\mathcal{T}_1$ that also contain x. So, we must have either $f^{-1}(\mathcal{B})\subseteq\mathcal{A}$ or, $\mathcal{A}\subseteq f^{-1}(\mathcal{B})$ which is possible only if $\mathcal{A}=f^{-1}(\mathcal{B})$ which gives $f(\mathcal{A})=\mathcal{B}$.

Proposition 5.2.5

For two A-TSs $(\mathcal{X},\mathcal{T}_1)$ and $(\mathcal{Y},\mathcal{T}_2)$, a map f from $\mathcal{T}_1$ to $\mathcal{T}_2$ is anti-continuous iff $(f^{-1}(\mathcal{B}))^{Anti-cl}\subseteq f^{-1}(\mathcal{B}^{Anti-cl})$ for each A-C subset $\mathcal{B}$ of $\mathcal{Y}$.

Proof:

Assume f to be anti-continuous, then $\mathcal{B}^{Anti-cl}$ is *A-C* with respect to $\mathcal{T}_2$ and so by ***definition 1.6.25***, $f^{-1}(\mathcal{B}^{Anti-cl})$ is *A-C* with respect to $\mathcal{T}_1$ and hence $[f^{-1}(\mathcal{B}^{Anti-cl})]^{Anti-cl}=f^{-1}(\mathcal{B}^{Anti-cl})$.

Now, $\mathcal{B}\subseteq\mathcal{B}^{Anti-cl}$ and so, $f^{-1}(\mathcal{B})\subseteq f^{-1}(\mathcal{B}^{Anti-cl})$

$\Rightarrow (f^{-1}(\mathcal{B}))^{Anti-cl}\subseteq[f^{-1}(\mathcal{B}^{Anti-cl})]^{Anti-cl}$, [by ***proposition 4.3.3 (iii)***]

But $[f^{-1}(\mathcal{B}^{Anti-cl})]^{Nt-cl}=f^{-1}(\mathcal{B}^{Anti-cl})$, so $(f^{-1}(\mathcal{B}))^{Anti-cl}\subseteq f^{-1}(\mathcal{B}^{Anti-cl})$.

Conversely, let the condition hold and let $\mathcal{C}$ be any *A-CS* with respect to $\mathcal{T}_2$ so that $\mathcal{C}^{Anti-cl} = \mathcal{C}$. Now, by condition $(f^{-1}(\mathcal{C}))^{Anti-cl} \subseteq f^{-1}(\mathcal{C}^{Anti-cl}) = f^{-1}(\mathcal{C})$

That is, $(f^{-1}(\mathcal{C}))^{Anti-cl} \subseteq f^{-1}(\mathcal{C})$.

But, $f^{-1}(\mathcal{C}) \subseteq (f^{-1}(\mathcal{C}))^{Anti-cl}$, [by ***proposition 4.3.3 (i)***]

Thus $(f^{-1}(\mathcal{C}))^{Anti-cl} = f^{-1}(\mathcal{C})$, thereby showing $f^{-1}(\mathcal{C})$ to be *A-C* with respect to $\mathcal{T}_1$ and hence as per ***definition 1.6.25***, the map f is anti-continuous.

Remark 5.2.2

The above proposition does not hold if the subset $\mathcal{B}$ of $\mathcal{Y}$ is not an *A-CS.*

Proposition 5.2.6

For two A-TSs $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is anti-continuous iff* $f(\mathcal{C}^{Anti-cl}) \subseteq [f(\mathcal{C})]^{Anti-cl}$ *for any A-C subset* $\mathcal{C}$ *of* $\mathcal{X}$.

Proof:

Assume f to be anti-continuous and $\mathcal{C}$ is some *A-C* subset of $\mathcal{X}$ and let $f(\mathcal{C}) = \mathcal{B} \subseteq \mathcal{Y}$.

Then ***proposition 5.2.6*** gives $(f^{-1}(\mathcal{B}))^{Anti-cl} \subseteq f^{-1}(\mathcal{B}^{Anti-cl})$

$$\Rightarrow [f^{-1}(\eta(\mathcal{C}))]^{Anti-cl} \subseteq f^{-1}[f(\mathcal{C}))^{Anti-cl}]$$

$\Rightarrow f^{-1}(f(\mathcal{C}^{Anti-cl})) \subseteq f^{-1}[(f(\mathcal{C}))^{Anti-cl}]$, since $f(\mathcal{C}) = f(\mathcal{C}^{Anti-cl})$

Thus, $f(\mathcal{C}^{Anti-cl}) \subseteq [f(\mathcal{C})]^{Anti-cl}$

Conversely, let the condition hold and assume $\mathcal{B}$ to be some *A-CS* with respect to $\mathcal{T}_2$, then $f^{-1}(\mathcal{B}) \subseteq \mathcal{X}$.

Now, by the condition, we have $f((f^{-1}(\mathcal{B}))^{Anti-cl}) \subseteq [f(f^{-1}(\mathcal{B}))]^{Anti-cl}$

$$\Rightarrow f((f^{-1}(\mathcal{B}))^{Anti-cl}) \subseteq f(f^{-1}(\mathcal{B}^{Anti-cl}))$$

$$\Rightarrow (f^{-1}(\mathcal{B}))^{Anti-cl} \subseteq f^{-1}(\mathcal{B}), \text{ given } \mathcal{B} \text{ is } A\text{-}C$$

But, $f^{-1}(\mathcal{B}) \subseteq (f^{-1}(\mathcal{B}))^{Anti-cl}$, [by ***proposition 4.3.3 (i)***]

Hence, we get $(f^{-1}(\mathcal{B}))^{Anti-cl} = f^{-1}(\mathcal{B})$ thereby showing $f^{-1}(\mathcal{B})$ to be *A-C* with respect to $\mathcal{T}_1$ and hence as per ***definition 1.6.25***, the function f is anti-continuous.

Proposition 5.2.7

For two A-TSs $(\mathcal{X},\mathcal{T}_1)$ *and* $(\mathcal{Y},\mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is anti-continuous if* $f^{-1}(\mathcal{A}^{Anti-int}) = [f^{-1}(\mathcal{A})]^{Anti-int}$ *for any A-O subset* $\mathcal{A}$ *of* $\mathcal{Y}$.

Proof:

Let the condition hold and assume that $\mathcal{B}$ is some *A-OS* in $\mathcal{T}_2$ so that $\mathcal{B}^{Anti-int} = \mathcal{B}$, then by the given condition we have:

$f^{-1}(\mathcal{B}^{Anti-int}) = [f^{-1}(\mathcal{B})]^{Anti-int}$,

Or, $f^{-1}(\mathcal{B}) = [f^{-1}(\mathcal{B})]^{Anti-int}$

This shows that $f^{-1}(\mathcal{B})$ is *A-O* in $\mathcal{T}_1$ and hence as per ***definition 1.6.25*** the map f is anti-continuous.

Definition 5.2.1

For two A-TSs $(\mathcal{X}, \mathcal{T}_1)$ *and* $(\mathcal{Y}, \mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is termed an A-O map if image of any* $\mathcal{T}_1$*-A-OS is* $\mathcal{T}_2$*-A-OS.*

Definition 5.2.2

For two A-TSs $(\mathcal{X}, \mathcal{T}_1)$ *and* $(\mathcal{Y}, \mathcal{T}_2)$, *a map* f *from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is called an A-C map if image of any* $\mathcal{T}_1$*-A-CS is* $\mathcal{T}_2$*-A-CS.*